A Logic of Plausible Justifications

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Abstract. In this work, we combine the frameworks of Justification Logics and Logics of Plausibility-Based Beliefs to build a logic for Multi-Agent Systems where each agent can explicitly state his justification for believing in a given sentence. Our logic is a normal modal logic based on the standard Kripke semantics, where we provide a semantic definition for the evidence terms and define the notion of plausible evidence for an agent, based on plausibility relations in the model. This way, unlike traditional Justification Logics, justifications can be actually faulty and unreliable. In our logic, agents can disagree not only over whether a sentence is true or false, but also on whether some evidence is a valid justification for a sentence or not. After defining our logic and its semantics, we provide a strongly complete axiomatic system for it and show that it has the finite model property and is decidable. Thus, this logic seems to be a good first step for the development of a dynamic logic that can model the processes of argumentation and debate in multi-agent systems.

1 Introduction and Motivation

Epistemic logics [15] are a particular kind of modal logics [10] where the modalities are used to describe epistemic notions such as *knowledge* and *belief* of agents. Traditional epistemic logics are expressive enough to describe knowledge and belief of multiple agents in a multi-agent system, including higher-order notions, such as the knowledge of one agent about the knowledge of another, and some notions of knowledge and belief that are related to groups of agents, such as "everybody in a group knows..." or "it is common knowledge in a group...".

Nevertheless, such epistemic logics have two important limitations. The first is that the knowledge or belief of an agent is static, i.e., it does not change over time. One of the reasons for this is that, in such logics, it is not possible to describe communication between the agents. The second is that the knowledge modeled by such logics is *implicit*, which means that if the agent knows something, then he knows it for some reason that remains unspecified.

In order to deal with the first limitation, *Dynamic Epistemic Logics* [11] were developed. In these logics, we can describe acts of communication between the agents. Such acts consist of *truthful* announcements that are made by one of the agents (or an external observer) to the other agents (or a sub-group of them).

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In works such as [4, 5, 7, 8], this framework of dynamic logics was extended so that not only knowledge, but also beliefs (which, unlike knowledge, may turn out to be actually false) could evolve over time. The semantics of such logics of *dynamic beliefs* is based on *Plausibility Models*, where each agent has a plausibility order for the possible states of the model and he believes in those sentences that are true in the most plausible states according to his plausibility order. The change of beliefs is then modeled as changes in the plausibility orders of the agents.

In order to deal with the second limitation, *Justification Logics* [1–3] were developed. In these logics, instead of formulas simply stating "Agent i knows φ", we have formulas that state "t is agent i's justification (or evidence) for knowing φ". Thus, these are logics of *explicit* knowledge, where every knowledge that an agent has is accompanied by an explicit justification for it. This is why Justification Logics are also called *Logics of Explicit Knowledge* or *Logics of Evidence-Based Knowledge*.

Justification Logics came from a previous framework, called Logic of Proofs [1], where justifications described formal proofs of arithmetical theorems. Thus, justifications in traditional Justification Logics are usually rather strong and the presence of a justification for a logical sentence entails the truth of that sentence [2, 3].

In the processes of argumentation and debate, be it an internal debate or a public debate where each agent tries to convince an external observer of his particular point of view, it is unrealistic to say that all of the announcements are *truthful*. The realistic assumption is that the announcements are merely *sincere*, i.e., each agent believes in what he announces. However, in order to convince others of their belief, an agent should not only state what he believes in, but also *why* he believes in it. So, the appropriate logic to model these processes would be a *dynamic logic of evidence-based beliefs*. In order to build such a logic, we combine aspects of Justification Logics with Plausibility Models, while considering now, unlike traditional Justification Logics, that justifications can actually be faulty and unreliable (so they no longer entail truth). Using Plausibility Models, we give a notion of plausible evidence, or plausible justification, for an agent. So, if an agent has a plausible evidence for a sentence, then he will believe in that sentence, but, as the evidence can possibly be faulty, this belief has the possibility to be false.

In the present work, we take a first step in order to build such a logic for the description of the processes of argumentation and debate. We build a normal modal logic (for the definition of normal modal logics, [10] can be consulted) where we can describe the plausibility of evidences for all the different agents, give a sound and strongly complete axiomatic system for this logic and show that it has the finite model property and is decidable.

As our next step, we plan to build a dynamic logic of *explicit* beliefs, adding to the present logic the actions that would model the communications between agents during the processes of argumentation and debate. This is not a trivial task. The standard announcements that describe changes of knowledge [11], sometimes called *hard announcements*, are too strong for our needs, since they are required to be *truthful* and not merely *sincere* (using such announcements without respecting the requisite that the announced formula should be true can generate logical inconsistencies). On the other hand, the standard announcements that describe changes of beliefs, called *belief upgrades* or *soft announcements* [6] are too weak, since, even tough they are only required to be *sincere*, they still make the agents receiving the announcement start believing in it, regardless of their current beliefs. In our desired framework, the announcement of a sentence should be accompanied by a justification as to why the agent performing the announcement believes in it. Then, each agent receiving the announcement should judge by himself whether he should start believing or not in the announced sentence, based on his current beliefs both about what was announced and about the justification that was given.

There are, in the literature, works that combine aspects of Justification Logics and Dynamic Epistemic Logics. [19] developed the first proposal of a Justification Logic with communication between the agents. However, these communication actions were extremely simple. Later, the series of works [16–18] developed logics that add to Justification Logics a series of communication actions, some rather complex. However, those actions are all from the family of *hard announcements*, so they cannot be used for our purpose. Our combination of Justification Logics with explicit evidence terms and Plausibility Models and our use of evidence terms to model explicit *beliefs* instead of explicit *knowledge* seems to be a novel approach. [9] also developed a logic of evidence-based beliefs, but, unlike our logic, it has no explicit evidence terms in the language and some of the modalities are non-normal. Besides that, also unlike our logic, no complete proof system for that logic is presented.

The rest of this paper is organized as follows. In Section 2, we introduce the necessary concepts that are used as building blocks for our logic: Justification Logic and Plausibility Models. The language and semantics of our logic, called Logic of Plausible Justifications (LPJ), is presented in Section 3, where we also show that our logic has the finite model property and is decidable and present a sound and strongly complete axiomatic system for it. Finally, in Section 4, we state our final remarks and point out potential further developments, including the one which originally motivated this work: the construction of a dynamic logic that can model argumentation and debate in multi-agent systems.

2 Background Concepts

This section presents two important concepts for the construction of our logic: Justification Logic and Plausibility Models.

2.1 Justification Logic

In this section, we provide a brief account of Justification Logic. For more details, [1–3, 12] can be consulted.

Definition 1. *The language of basic Justification Logic consists of a countable set Φ of proposition symbols, a countable set $\mathcal{C}$ of* evidence constants*, a countable set $\mathcal{X}$ of* evidence variables*, all pairwise disjoint, and the boolean connectives $\neg$ and $\wedge$. The* formulas *φ and the* evidence terms *t of the language are defined as follows:*

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid t : \varphi, \text{with } t ::= c \mid x \mid t_1 \cdot t_2 \mid t_1 + t_2 \mid !t \ ,$$

where $p \in \Phi$, $c \in \mathcal{C}$ and $x \in \mathcal{X}$. We denote the set of all evidence terms of the language by $\mathcal{T}$ and the set of all formulas by F.

In this logic and in every other logic described in this paper, we use the standard abbreviations $\bot \equiv \neg\top$, $\varphi \vee \phi \equiv \neg(\neg\varphi \wedge \neg\phi)$, $\varphi \to \phi \equiv \neg(\varphi \wedge \neg\phi)$ and $\varphi \leftrightarrow \phi \equiv (\varphi \to \phi) \wedge (\phi \to \varphi)$.

Initially, the basic Justification Logic was defined in a purely syntactic manner, through an axiomatic system. Later, Fitting [12] provided a modal semantics for the logic.

Definition 2. *A* frame *for Justification Logic is a tuple $\mathcal{F} = (W, R)$ where W is a non-empty set of states and $R \subseteq W \times W$ is a binary relation that is* reflexive *and* transitive*.*

Definition 3. *A* Fitting Model *for Justification Logic is a tuple $\mathcal{M} = (\mathcal{F}, \mathbf{V}, \mathcal{E})$, where $\mathcal{F}$ is a frame, $\mathbf{V}$ is a valuation function $\mathbf{V} : \Phi \mapsto 2^W$ and $\mathcal{E}$ is an* evidence *function $\mathcal{E} : W \times \mathcal{T} \to F$ satisfying the following conditions:*

- *If $(\varphi \to \psi) \in \mathcal{E}(w, s)$ and $\varphi \in \mathcal{E}(w, t)$, then $\psi \in \mathcal{E}(w, s \cdot t)$.*
- *$\mathcal{E}(w, s) \cup \mathcal{E}(w, t) \subseteq \mathcal{E}(w, s + t)$.*
- *If $\varphi \in \mathcal{E}(w, t)$, then $t : \varphi \in \mathcal{E}(w, !t)$.*
- *If wRw', then $\mathcal{E}(w, t) \subseteq \mathcal{E}(w', t)$.*
- *If $\varphi \in \mathcal{E}(w, c)$ and $c \in \mathcal{C}$, then φ must be* valid*, as defined below.*

Definition 4. *Let $\mathcal{M} = (\mathcal{F}, \mathbf{V}, \mathcal{E})$ be a Fitting model. The notion of* satisfaction *of a formula φ in a model $\mathcal{M}$ at a state w, notation $\mathcal{M}, w \Vdash \varphi$, can be inductively defined as follows:*

- *$\mathcal{M}, w \Vdash p$ iff $w \in \mathbf{V}(p)$.*
- *$\mathcal{M}, w \Vdash \top$ always.*
- *$\mathcal{M}, w \Vdash \neg\varphi$ iff $\mathcal{M}, w \nVdash \varphi$.*
- *$\mathcal{M}, w \Vdash \varphi_1 \wedge \varphi_2$ iff $\mathcal{M}, w \Vdash \varphi_1$ and $\mathcal{M}, w \Vdash \varphi_2$.*
- *$\mathcal{M}, w \Vdash t : \varphi$ iff $\varphi \in \mathcal{E}(w, t)$ and, for all w' such that wRw', $\mathcal{M}, w \Vdash \varphi$.*

If $\mathcal{M}, w \Vdash \varphi$ for every state w, we say that φ is *globally satisfied* in the model $\mathcal{M}$, notation $\mathcal{M} \Vdash \varphi$. If φ is globally satisfied in all models $\mathcal{M}$ of a frame $\mathcal{F}$, we say that φ is *valid* in $\mathcal{F}$, notation $\mathcal{F} \Vdash \varphi$. Finally, if φ is valid in all frames, we say that φ is valid, notation $\Vdash \varphi$.

From the semantical definition above, we can think of $\cdot$ as a form of evidence-controlled Modus Ponens, $+$ as a form of evidence combination, ! as a constructor of evidence for formulas that already contain evidence terms and evidence constants as atomic evidence for formulas that do not need further justification (since they are valid).

2.2 Plausibility Models

In this section, we present Plausibility Models for the single-agent case. The multi-agent case is covered in the presentation of our logic in the next section. Plausibility Models in the present form were introduced in [4, 5] and [7, 8].

Definition 5. *A* Plausibility Frame *is a tuple $\mathcal{F} = (W, \geq)$, where W is a non-empty set of* states *and $\geq \subseteq W \times W$ is a relation that satisfies* reflexivity*,* transitivity *(thus, is a* pre-order*) and* local connectivity *(for all $v, w \in W$, $v \geq w$ or $w \geq v$ or both). $\geq$ is called a* plausibility order*.*

When we think about the relation $\geq$ as an epistemic relation, we consider that, if $v \geq w$, then the agent does not know for sure in which of the states v or w he actually is, but he considers that the state w is *more* plausible than the state v. The choice of the most plausible states as the minimal states according to the pre-order $\geq$, which seems counter-intuitive, comes from the fact that if we add the hypothesis that $\geq$ is well-founded, then we guarantee that the set of most plausible states is always well-defined.

As $\geq$ is transitive and locally connected, the agent considers that it is possible for him to be in any state of the model, but he considers some more plausible than others. In the particular case that $v \geq w$ and $w \geq v$, the agent considers both states v and w to be equally plausible.

Definition 6. *A* Plausibility Model *is a tuple $\mathcal{M} = (\mathcal{F}, \mathbf{V})$, where $\mathcal{F}$ is a Plausibility Frame and $\mathbf{V}$ is a* valuation function $\mathbf{V} : \Phi \to 2^W$*, mapping proposition symbols to sets of states.*

Let us now discuss the kinds of *belief* that can be described in a Plausibility Model. One thing that all sorts of beliefs have in common, and what differentiates them from *knowledge*, is that there is always the possibility that a belief can be false. Nevertheless, beliefs are *consistent*, which means that an agent cannot simultaneously believe in ϕ and $\neg\phi$.

For our discussion of beliefs, let us consider that the model $\mathcal{M}$ is finite or that the relation $\geq$ is well-founded. We can define the set $\text{Best}(\mathcal{M}) = \{w \in W : v \geq w, \text{ for all } v \in W\}$. Then, we can define the weakest notion of belief (denoted by $\mathcal{B}$) as $\mathcal{M}, w \Vdash \mathcal{B}\varphi$ iff $\mathcal{M}, v \Vdash \varphi$, for all $v \in \text{Best}(\mathcal{M})$. Thus, an agent believes in φ if the formula is satisfied in the most plausible states, according to his plausibility order. The modality $\mathcal{B}$ is a normal modality ($\mathcal{B}(\varphi \to \psi) \to (\mathcal{B}\varphi \to \mathcal{B}\psi)$). However, $\mathcal{B}$ is not the modality directly associated with the relation $\geq$. Let us then define $\mathcal{M}, w \Vdash \Box\varphi$ iff $\mathcal{M}, v \Vdash \varphi$, for all v such that $w \geq v$. The notion described by $\Box$ is called *safe belief*. An agent has safe belief in a formula if it is satisfied in all states that are more or equally plausible than the current one. Thus, safe belief implies belief. Safe belief is also normal. However, the "safety" of a belief can only be inferred by an external observer, because for an agent to know that one of his beliefs is safe he would need to know in which state he currently is. Finally, we define the notion of *strong belief* in a formula φ if all the states in which φ is satisfied are more plausible than all the states in which φ is not satisfied. Strong belief also implies belief, but strong belief is not normal.

3 Logic of Plausible Justifications

In this section, we present our logic, discuss some of its expressive features and present a sound and strongly complete axiomatic system for it.

3.1 Language and Semantics

We start by defining the language for the formulas of our logic.

Definition 7. *In order to define the language of LPJ, we need to take a finite set $\mathcal{A} = \{1, \ldots, n\}$ of agents, a countable set Φ of* proposition symbols *and countable sets $\mathcal{X}_i$, $i \in \mathcal{A}$, of* evidence variables. *We assume that each pair of such sets is disjoint. The* formulas φ *and the* evidence terms t *of the language are defined as follows:*

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathcal{K}_i\varphi \mid \Box_i\varphi \mid [t?]\varphi \mid t \gg_i \varphi \mid \mathcal{P}_i t \mid t :_i \varphi, \text{ with}$$

$$t ::= p \mid g \mid x^i_j \mid \bar{t}, \mid t_1 + t_2 \ ,$$

where $p \in \Phi$, $i \in \mathcal{A}$, $j \in \mathbb{N}$, $x^i_j \in \mathcal{X}_i$ and g is a term not occurring in $\mathcal{A}$, Φ or any of the sets $\mathcal{X}_i$. We denote the set of all evidence terms of the language by $\mathcal{T}$.

In the rest of this paper, sometimes it is convenient to use the duals of some of our modalities: $\langle\mathcal{K}_i\rangle\varphi \equiv \neg\mathcal{K}_i\neg\varphi$, $\Diamond_i\varphi \equiv \neg\Box_i\neg\varphi$ and $\langle t?\rangle\varphi \equiv \neg[t?]\neg\varphi$. We do not use the duals of $\gg_i$, $\mathcal{P}_i$ and $:_i$, but they can be defined analogously.

The first thing that we notice is that there are some differences between our evidence terms and the ones in Justification Logic. Our language does not have the operators $\cdot$ and $!$, it does not have a set $\mathcal{C}$ of evidence constants, having instead a single evidence constant g, it has evidence terms of the form $\bar{t}$ and proposition symbols are also evidence terms. We will discuss these differences after we present the semantics of our logic.

Definition 8. *A* frame *for LPJ is a tuple $\mathcal{F} = (W, \{\sim_i\}_{i\in\mathcal{A}}, \{\geq_i\}_{i\in\mathcal{A}})$ where*

- *W is a non-empty set of* states
- *$\sim_i \subseteq W \times W$ is an* equivalence relation.
- *$\geq_i \subseteq W \times W$ is a relation that satisfies* reflexivity *and* transitivity.
- *For each $i \in \mathcal{A}$, the relations $\sim_i$ and $\geq_i$ satisfy the following property: $\sim_i = \geq_i \cup (\geq_i)^{-1}$, where $(\geq_i)^{-1} = \{(v, w) \in W \times W : (w, v) \in \geq_i\}$.*
- *Building the relation $\approx = (\bigcup_{i\in\mathcal{A}} \sim_i)$, we have that, for every pair $(v, w) \in W \times W$, $v \approx w$. We call this property* weak connectivity.

In an LPJ frame, the relations $\geq_i$ are the plausibility relations for each of the agents. They are pre-orders, just as in the single-agent setting of the previous section. Also as in the previous section, if $v \geq_i w$, then agent i does not know for sure in which of the states v or w he actually is, but he considers state w *more* plausible than state v. The relations $\sim_i$ denote these relations of indistinguishability of states by each of the agents. This is why they are defined as $\sim_i = \geq_i \cup (\geq_i)^{-1}$.

The relations $\geq_i$ are no longer locally connected in our present multi-agent scenario. However, the property of weak connectivity implies that every pair of states in the frame is indistinguishable to at least one of the agents.

Definition 9. *A* model *for LPJ is a tuple $\mathcal{M} = (\mathcal{F}, \mathbf{V}, \mathcal{E})$, where $\mathcal{F}$ is a frame, $\mathbf{V}$ is a* valuation function $\mathbf{V} : \Phi \to 2^W$, *mapping proposition symbols to sets of states, and $\mathcal{E}$ is an* evidence function $\mathcal{E} : \mathcal{T} \to 2^W$, *mapping evidence terms to sets of states, that satisfies the following rules:*

- $\mathcal{E}(p) = \mathbf{V}(p)$, *for all* $p \in \Phi$.
- $\mathcal{E}(g) = W$.
- $\mathcal{E}(\overline{t}) = W \setminus \mathcal{E}(t)$.
- $\mathcal{E}(t_1 + t_2) = \mathcal{E}(t_1) \cap \mathcal{E}(t_2)$.

We can see, in this definition, how the semantics for our evidence terms is built. We follow a similar approach to the one presented in [9] and use what seems to be the simplest semantics for evidence: an evidence is a subset of states of the model. Roughly speaking, we say that an evidence term t is an evidence for a formula φ if φ is satisfied in all of the states in $\mathcal{E}(t)$ (in reality, we also have to check the *plausibility* of the evidence, as we discuss below).

As the proposition symbols also denote subsets of states (using the function $\mathbf{V}$), we also use them as evidence terms. We can think of the evidences denoted by proposition symbols as the "common ground" for all of the agents. Then, we can use the variables in the sets $\mathcal{X}_i$ to denote the "personal views" of each agent. The evidence term g can be considered as the "weakest" evidence, as it contains all the states in the model. It has a similar function to the evidence constants in Justification Logic (looking at the semantics below, we can see that g can be used by any agent as evidence for what he *knows*) and it can also replace the operator !, as we can see, from the definition below, that the formula $t :_i \varphi \to g :_i (t :_i \varphi)$ is valid. We can drop the operator $\cdot$ since, in our semantics, it would be a particular case of the operator $+$ to formulas in implication form. Finally, evidence terms of the form $\overline{t}$ denote evidence *complementation*.

Definition 10. *Let $\mathcal{M} = (\mathcal{F}, \mathbf{V}, \mathcal{E})$ be a model. The notion of* satisfaction *of a formula φ in a model $\mathcal{M}$ at a state w, notation $\mathcal{M}, w \Vdash \varphi$, can be inductively defined as follows:*

- $\mathcal{M}, w \Vdash p$ *iff* $w \in \mathbf{V}(p)$.
- $\mathcal{M}, w \Vdash \top$ *always.*
- $\mathcal{M}, w \Vdash \neg\varphi$ *iff* $\mathcal{M}, w \nVdash \varphi$.
- $\mathcal{M}, w \Vdash \varphi_1 \wedge \varphi_2$ *iff* $\mathcal{M}, w \Vdash \varphi_1$ *and* $\mathcal{M}, w \Vdash \varphi_2$.
- $\mathcal{M}, w \Vdash \mathcal{K}_i\varphi$ *iff for all* $v \in W$ *such that* $w \sim_i v$, $\mathcal{M}, v \Vdash \varphi$.
- $\mathcal{M}, w \Vdash \Box_i\varphi$ *iff for all* $v \in W$ *such that* $w \geq_i v$, $\mathcal{M}, v \Vdash \varphi$.
- $\mathcal{M}, w \Vdash [t?]\varphi$ *iff if* $w \in \mathcal{E}(t)$, *then* $\mathcal{M}, w \Vdash \varphi$.
- $\mathcal{M}, w \Vdash t \gg_i \varphi$ *iff for all* $v \in W$ *such that* $w \sim_i v$ *and* $v \in \mathcal{E}(t)$, $\mathcal{M}, v \Vdash \varphi$.
- $\mathcal{M}, w \Vdash \mathcal{P}_i t$ *iff:*
  1. *there is* $v \in W$ *such that* $w \sim_i v$ *and* $v \in \mathcal{E}(t)$ *and*

2. *for all* $x, y \in W$ *such that* $w \sim_i x$ *and* $x \geq_i y$*, if* $x \in \mathcal{E}(t)$*, then* $y \in \mathcal{E}(t)$*.*

– $\mathcal{M}, w \Vdash t :_i \varphi$ *iff* $\mathcal{M}, w \Vdash t \gg_i \varphi$ *and* $\mathcal{M}, w \Vdash \mathcal{P}_i t$*.*

The notions of global satisfaction, validity in a frame and validity are defined as in the previous section. We say that a formula φ is *satisfiable* if there is a model $\mathcal{M}$ and a state w in $\mathcal{M}$ such that $\mathcal{M}, w \Vdash \varphi$. A formula is not satisfiable iff its negation is valid. A (possibly infinite) set Δ of formulas is satisfiable if there is a *single model* $\mathcal{M}$ and a *single state* w in $\mathcal{M}$ such that $\mathcal{M}, w \Vdash \varphi$, for all $\varphi \in \Delta$.

Our modalities $\mathcal{K}_i$, $\Box_i$, $[t?]$, $\gg_i$ and $:_i$ are all normal. The modalities $\mathcal{K}_i$ and $\Box_i$ denote the usual notions of *knowledge* (satisfaction in all states indistinguishable from the current one) and *safe belief* (satisfaction in all states more or equally plausible than the current one), respectively.

Our modalities $[t?]$ are inspired by PDL [14] test modalities. They allow us to verify whether a state belongs to an evidence t, by checking whether $\langle t? \rangle \top$ is satisfied at a state. The relation associated to the modality $[t?]$ is $R_t = \{(w, w) : w \in \mathcal{E}(t)\}$. The modalities $\gg_i$ are inspired by Renne's [16–18] modality of *admissibility*. The semantics of a formula $t : \varphi$ in Justification Logic is composed by two parts (see Section 2.1): the first one related to the evidence function and the second to the relations in the frame. Renne calls the first part *admissibility* and uses the modality $\gg$ to describe it. In our semantics, an agent i considers an evidence t *admissible* for a formula φ ($t \gg_i \varphi$) if, in all the states inside the evidence t that the agent consider possible for him to be, the formula φ is satisfied.

The modalities $\mathcal{P}_i$ are used to denote that agent i considers an evidence to be *plausible*. The notion of *plausibility* of an evidence has some similarities to the notion of *strong belief*. An evidence is considered plausible if there is a state that the agent considers possible inside of the evidence and, among the states that the agent considers possible, all of the states inside the evidence are more plausible than all of the states outside the evidence. Finally, the modality $:_i$ is used to denote that an agent considers an evidence as *plausible evidence* or *plausible justification* for a formula. An evidence is plausible evidence for a formula if the agent considers the evidence to be plausible and considers the evidence to be admissible for the formula.

Theorem 1 (Finite Model Property). *Every satisfiable formula* φ *is satisfiable in a finite model (i.e., a model with a finite number of states).*

Proof. See appendix A. □

Corollary 1 (Decidability). *The satisfiability problem for LPJ (determining whether a formula* φ *is satisfiable) is decidable.*

Proof. The proof of Theorem 1 gives an upper bound (as a function of the size of φ) for the size of the models where φ must be checked. As there is a finite number of such models, they can all be verified to determine whether φ is satisfiable. □

3.2 Axiomatic System

We consider the set of axioms and rules in Figure 1, where φ and ψ are arbitrary formulas, t and s are arbitrary evidence terms and p is an arbitrary proposition symbol. We present the axioms divided in groups related to their function.

1. Tautologies, Duals and Normality
   PL Propositional Tautologies
   $\mathbf{Du}_{\mathcal{K}}$ $\mathcal{K}_i\varphi \leftrightarrow \neg\langle\mathcal{K}_i\rangle\neg\varphi$
   $\mathbf{Du}_{\Box}$ $\Box_i\varphi \leftrightarrow \neg\Diamond_i\neg\varphi$
   $\mathbf{Du}_t$ $[t?]\varphi \leftrightarrow \neg\langle t?\rangle\neg\varphi$
   $\mathbf{K}_{\mathcal{K}}$ $\mathcal{K}_i(\varphi \rightarrow \psi) \rightarrow (\mathcal{K}_i\varphi \rightarrow \mathcal{K}_i\psi)$
   $\mathbf{K}_{\Box}$ $\Box_i(\varphi \rightarrow \psi) \rightarrow (\Box_i\varphi \rightarrow \Box_i\psi)$
   $\mathbf{K}_t$ $[t?](\varphi \rightarrow \psi) \rightarrow ([t?]\varphi \rightarrow [t?]\psi)$
2. $\sim_i$ is an equivalence relation
   $\mathbf{T}_{\mathcal{K}}$ $\mathcal{K}_i\varphi \rightarrow \varphi$
   $\mathbf{4}_{\mathcal{K}}$ $\mathcal{K}_i\varphi \rightarrow \mathcal{K}_i\mathcal{K}_i\varphi$
   $\mathbf{5}_{\mathcal{K}}$ $\neg\mathcal{K}_i\varphi \rightarrow \mathcal{K}_i\neg\mathcal{K}_i\varphi$
3. $\geq_i$ is reflexive and transitive
   $\mathbf{T}_{\Box}$ $\Box_i\varphi \rightarrow \varphi$
   $\mathbf{4}_{\Box}$ $\Box_i\varphi \rightarrow \Box_i\Box_i\varphi$
4. $\sim_i = \geq_i \cup (\geq_i)^{-1}$
   $\mathbf{Rel}_1$ $\mathcal{K}_i\varphi \rightarrow \Box_i\varphi$
   $\mathbf{Rel}_2$ $\langle\mathcal{K}_i\rangle\varphi \wedge \langle\mathcal{K}_i\rangle\psi \rightarrow \langle\mathcal{K}_i\rangle(\varphi \wedge \Diamond\psi) \vee \langle\mathcal{K}_i\rangle(\Diamond\varphi \wedge \psi)$
5. Construction of evidence terms
   $\mathbf{E}_1$ $\langle t?\rangle\varphi \leftrightarrow (\langle t?\rangle\top \wedge \varphi)$
   $\mathbf{E}_2$ $\langle p?\rangle\top \leftrightarrow p$
   $\mathbf{E}_3$ $\langle g?\rangle\top$
   $\mathbf{E}_4$ $\langle \bar{t}?\rangle\top \leftrightarrow \neg\langle t?\rangle\top$
   $\mathbf{E}_5$ $(\langle t?\rangle\top \wedge \langle s?\rangle\top) \leftrightarrow \langle(s+t)?\rangle\top$
6. Admissibility, Plausibility and Justification
   Adm $(t \gg_i \varphi) \leftrightarrow (\mathcal{K}_i[t?]\varphi)$
   Pla $\mathcal{P}_i t \leftrightarrow ((\mathcal{K}_i[t?]\Box_i\langle t?\rangle\top) \wedge (\langle\mathcal{K}_i\rangle\langle t?\rangle\top))$
   Jus $t :_i \varphi \leftrightarrow (t \gg_i \varphi) \wedge \mathcal{P}_i t$
7. Rules
   MP From $\varphi \rightarrow \psi$ and φ, derive ψ
   Gen From φ, derive $\mathcal{K}_i\varphi$, $\Box_i\varphi$ and $[t?]\varphi$

Fig. 1. Axiomatic System

From the axioms above, perhaps $\mathbf{Rel}_2$ is the one with the less clear purpose. $\mathbf{Rel}_1$ states that every pair in $\geq_i$ is also in $\sim_i$, while $\mathbf{Rel}_2$ states that every pair in $\sim_i$ is in $\geq_i$ or in $(\geq_i)^{-1}$. $\mathbf{Rel}_2$ is an adaptation with two modalities ($\langle\mathcal{K}_i\rangle$ and $\Diamond_i$) of the so-called **.3** axiom (see [10] for more details about this axiom).

Every formula ϕ derivable from the axiomatic system above is called a *theorem* (denoted by $\vdash \phi$). A formula ϕ is *consistent* iff $\neg\phi$ is not a theorem, i.e., iff $\nvdash \neg\phi$, and *inconsistent* otherwise. A finite set of formulas $\Delta = \{\phi_1, \ldots, \phi_n\}$ is consistent iff the formula $\psi = \phi_1 \wedge \ldots \wedge \phi_n$ is consistent. Finally, an infinite set of formulas Δ' is consistent iff every finite subset $\Delta \subset \Delta'$ is consistent.

The axiomatic system is said to be *sound* if every satisfiable formula is consistent (or, in an equivalent definition, if every satisfiable set of formulas is consistent). The axiomatic system is said to be *complete* if every consistent formula is satisfiable. It is said to be *strongly complete* if every consistent set of formulas is satisfiable. Unlike the case of soundness, the two definitions for completeness are not equivalent. Strong completeness implies completeness, but the reciprocal is false.

The proof of the soundness of our axiomatic system is straightforward. It is not difficult to show that each of the axioms is valid according to the LPJ semantics and that the application of each of the rules to valid formulas give formulas that are also valid. The strong completeness proof is given in the following theorem.

Theorem 2 (Strong Completeness). *Every consistent set of formulas is satisfiable in an LPJ model.*

Proof. See appendix B. □

4 Final Remarks and Future Work

In this work, we combine aspects of Justification Logics with Plausibility Models to build a logic of *explicit* beliefs, where each agent can explicitly state which is his justification for believing in a given sentence. Our logic is a normal modal logic based on the standard Kripke semantics, where we provide a semantic definition for the evidence terms and define the notion of plausible evidence for an agent, based on plausibility relations in the model. In our logic, agents can disagree not only over whether a sentence is true or false, but also on whether some evidence is a valid justification for a sentence or not. Thus, unlike traditional Justification Logics, justifications can be faulty and unreliable in our logic. After defining our logic and its semantics, we provide a strongly complete axiomatic system for it and show that it has the finite model property and is decidable.

We feel that this logic is a good first step for the development of a dynamic logic that can model the processes of argumentation and debate in multi-agent systems. We think that the appropriate logic to model these processes would be a *dynamic* logic of evidence-based beliefs. Thus, as our next step to build such a logic, we need to add to the present logic the actions that would model the communications between agents during the processes of argumentation and debate. In our desired framework, the announcement of a sentence should be accompanied by a justification as to why the agent performing the announcement believes in it. Then, each agent receiving the announcement should judge by himself whether he should start believing or not in the announced sentence, based on his current beliefs both about what was announced and about the justification that was given. In a preliminary analysis, it seems that we may have a few possible results for an announcement, depending on whether the agent receiving the announcement currently (before the announcement) believes in what was announced, in the negation of it or in neither and whether he currently considers that the justification that was given is plausible or not.

Beside this main goal, we feel that it would also be interesting to develop other proof systems for this logic, such as a tableau system or a sequent calculus, since they are more suited to be used as automatic provers than an axiomatic system. It would also be interesting to investigate the model-checking problem for this logic and to analyze its complexity. Finally, it would be interesting to analyze extensions of this logic with quantification over evidence terms, as [13] did in the context of traditional Justification Logics.

References

1. Artemov, S.: Logic of proofs. Annals of Pure and Applied Logic 67(1–3), 29–59 (1994)
2. Artemov, S.: Justified common knowledge. Theoretical Computer Science 357(1–3), 4–22 (2006)
3. Artemov, S., Nogina, E.: Introducing justification into epistemic logic. Journal of Logic and Computation 15(6), 1059–1073 (2005)
4. Baltag, A., Smets, S.: Conditional doxastic models: A qualitative approach to dynamic belief revision. In: Queiroz, R., Mints, G. (eds.) Proceedings of the 13th Workshop on Logic, Language, Information and Computation (WoLLIC 2006). Electronic Notes in Theoretical Computer Science, vol. 165, pp. 5–21. Elsevier, Amsterdam (2006)
5. Baltag, A., Smets, S.: Dynamic belief revision over multi-agent plausibility models. In: Bonano, G., van der Hoek, W., Wooldridge, M. (eds.) Proceedings of the 7th Conference on Logic and the Foundations of Game and Decision Theory (LOFT 2006). pp. 11–24. Liverpool (2006)
6. Baltag, A., Smets, S.: Talking your way into agreement: Belief merge by persuasive communication. In: Baldoni, M. et al. (ed.) Proceedings of the Second Multi-Agent Logics, Languages, and Organisations Federated Workshops. CEUR Workshop Proceedings, vol. 494, pp. 129–141. CEUR-WS.org, Aachen (2009), `http://ceur-ws.org/Vol-494/`
7. van Benthem, J.: Dynamic logic for belief revision. Journal of Applied Non-Classical Logics 17(2), 129–155 (2007)
8. van Benthem, J., Liu, F.: Dynamic logic of preference upgrade. Journal of Applied Non-Classical Logics 17(2), 157–182 (2007)
9. van Benthem, J., Pacuit, E.: Dynamic logics of evidence-based beliefs. Studia Logica 99(1-3), 61–92 (2011)
10. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic, Cambridge Tracts in Theoretical Computer Science, vol. 53. Cambridge University Press, Cambridge (2001)
11. van Ditmarsch, H., van der Hoek, W., Kooi, B.: Dynamic Epistemic Logic. Synthese Library, Springer, Heidelberg (2007)
12. Fitting, M.: The logic of proofs, semantically. Annals of Pure and Applied Logic 132(1), 1–25 (2005)
13. Fitting, M.: A quantified logic of evidence. Annals of Pure and Applied Logic 152(1-3), 67–83 (2008)
14. Harel, D., Kozen, D., Tiuryn, J.: Dynamic Logic. Foundations of Computing, MIT Press, Cambridge (2000)
15. van der Hoek, W., Verbrugge, R.: Epistemic logic: a survey. In: Petrosjan, L.A., Mazalov, V.V. (eds.) Game Theory and Applications, vol. 8, pp. 53–94. Nova Science Publishers, New York (2002)
16. Renne, B.: Dynamic Epistemic Logic with Justification. Ph.D. thesis, The City University of New York (2008)
17. Renne, B.: Public communication in justification logic. Journal of Logic and Computation 21(6), 1005–1034 (2011)
18. Renne, B.: Multi-agent justification logic: Communication and evidence elimination. Synthese 185(S1), 43–82 (2012)
19. Yavorskaya, T.: Interacting explicit evidence systems. Theory of Computing Systems 43(2), 272–293 (2008)

A Finite Model Property

In this section, we present a proof of the finite model property for our logic using the standard technique of *filtrations*, adapting it to the particular features of our logic. For more details on this construction, [10] can be consulted.

Definition 11. *Given a formula φ of the language, we built the set of formulas $Cl(\varphi)$, called* closure *of φ, which is the smallest set that contains φ and satisfies the following properties:*

1. *If $\psi \in Cl(\varphi)$, then, for all subformulas ψ' of ψ, $\psi' \in Cl(\varphi)$.*
2. *If $\blacksquare\psi \in Cl(\varphi)$, then $\neg\blacklozenge\neg\psi \in Cl(\varphi)$, where $\blacksquare \in \{\mathcal{K}_i, \Box_i, [t?]\}$ and $\blacklozenge$ is the corresponding dual.*
3. *If $t \gg_i \psi \in Cl(\varphi)$, then $\mathcal{K}_i[t?]\psi \in Cl(\varphi)$.*
4. *If $\mathcal{P}_i t \in Cl(\varphi)$, then $\mathcal{K}_i[t?]\Box_i\langle t?\rangle\top \in Cl(\varphi)$ and $\langle\mathcal{K}_i\rangle\langle t?\rangle\top \in Cl(\varphi)$.*
5. *If $t :_i \psi \in Cl(\varphi)$, then $t \gg_i \psi \in Cl(\varphi)$ and $\mathcal{P}_i t \in Cl(\varphi)$.*
6. *If $\langle t?\rangle\psi \in Cl(\varphi)$, then $\langle t?\rangle\top \in Cl(\varphi)$.*

It is important to notice that $Cl(\varphi)$ is a finite set. Using this set of formulas, we can define an equivalence relation in the set of states of an LPJ model $\mathcal{M} = (W, \{\sim_i\}_{i\in\mathcal{A}}, \{\geq_i\}_{i\in\mathcal{A}}, \mathbf{V}, \mathcal{E})$. We define $w \leftrightsquigarrow w'$ if, for all formulas $\psi \in Cl(\varphi)$, $\mathcal{M}, w \Vdash \psi$ iff $\mathcal{M}, w' \Vdash \psi$. We denote the equivalence class of state w by this equivalence relation as $|w|$. We can then use this equivalence relation to build a new model $\mathcal{M}^f$ from $\mathcal{M}$.

Definition 12. *Let $\mathcal{M}^f = (W^f, \{\sim_i^f\}_{i\in\mathcal{A}}, \{\geq_i^f\}_{i\in\mathcal{A}}, \mathbf{V}^f, \mathcal{E}^f)$, where:*

- *$W^f = \{|w| : w \in W\}$.*
- *$|w| \sim_i^f |v|$ iff, for all formulas $\langle\mathcal{K}_i\rangle\psi \in Cl(\varphi)$, $\mathcal{M}, w \Vdash \langle\mathcal{K}_i\rangle\psi$ iff $\mathcal{M}, v \Vdash \langle\mathcal{K}_i\rangle\psi$.*
- *$|w| \geq_i^f |v|$ iff, for all formulas $\Diamond_i\psi \in Cl(\varphi)$, $\mathcal{M}, v \Vdash \Diamond_i\psi$ implies that $\mathcal{M}, w \Vdash \Diamond_i\psi$.*
- *$\mathbf{V}^f(p) = \{|w| \in W^f : w \in \mathbf{V}(p)\}$, for all proposition symbols $p \in Cl(\varphi)$.*
- *$\mathcal{E}^f(t) = \{|w| \in W^f : w \in \mathcal{E}(t)\}$, for all evidence terms t such that $\langle t?\rangle\top \in Cl(\varphi)$.*

It is straightforward to verify that the relations $\sim_i^f$ and $\geq_i^f$ satisfy the necessary conditions stated in [10] for filtration relations and also to check that $\mathcal{M}^f$ indeed satisfies all the properties of an LPJ model: $\sim_i^f$ is an equivalence relation, $\geq_i^f$ is reflexive and transitive, we have $\sim_i^f = \geq_i^f \cup (\geq_i^f)^{-1}$, $\mathcal{M}^f$ is weakly connected and the evidence function satisfies the desired inductive rules. We call the model $\mathcal{M}^f$ the filtration of $\mathcal{M}$ through $Cl(\varphi)$. It is important to notice that, if $|Cl(\varphi)| = k$, then $|W^f| = 2^k$, so $\mathcal{M}^f$ is a *finite model.*

Theorem 3 (Filtration Theorem). *For all formulas $\psi \in Cl(\varphi)$ and all states w in $\mathcal{M}$, $\mathcal{M}, w \Vdash \psi$ iff $\mathcal{M}^f, |w| \Vdash \psi$.*

Proof. The proof is by induction on the structure of the formula ψ.

– If ψ is a proposition symbol, $\psi = \top$, $\psi = \neg\psi_1$ or $\psi = \psi_1 \wedge \psi_2$, the proof is straightforward from the definition of $\mathbf{V}^f$ and from item 1 of definition 11.
– If $\psi = \mathcal{K}_i\psi_1$, then, by items 1 and 2 of definition 11, $\phi = \langle\mathcal{K}_i\rangle\phi_1 \in Cl(\varphi)$, where $\phi_1 = \neg\psi_1$. Then, using the induction step for negation, it is sufficient to show the result for ϕ.
  (⇒) Suppose that $\mathcal{M}, w \Vdash \langle\mathcal{K}_i\rangle\phi_1 \in \Gamma$. Then, there is v in $\mathcal{M}$ such that $w \sim_i v$ and $\mathcal{M}, v \Vdash \phi_1$. By the induction hypothesis, $\mathcal{M}^f, |v| \Vdash \phi_1$. Now, as $\sim_i^f$ satisfies the necessary conditions for filtration relations, we have that $w \sim_i v$ implies $|w| \sim_i^f |v|$. Thus, $\mathcal{M}^f, |w| \Vdash \langle\mathcal{K}_i\rangle\phi_1$.
  (⇐) Suppose that $\mathcal{M}^f, |w| \Vdash \langle\mathcal{K}_i\rangle\phi_1$. Then, there exists $|v|$ in $\mathcal{M}^f$ such that $|w| \sim_i^f |v|$ and $\mathcal{M}, |v| \Vdash \phi_1$. By the induction hypothesis, $\mathcal{M}, v \Vdash \phi_1$. Now, as $\sim_i^f$ satisfies the necessary conditions for filtration relations, we have that $|w| \sim_i^f |v|$ and $\mathcal{M}, v \Vdash \phi_1$ implies that $\mathcal{M}, w \Vdash \langle\mathcal{K}_i\rangle\phi_1$.
– If $\psi = \Box_i\psi_1$, the proof is entirely analogous to the previous item.
– If $\psi = [t?]\psi_1$, then, using the same reasoning of the previous two cases, it is sufficient to show the result for formulas of the form $\phi = \langle t?\rangle\phi_1$. Now, $\mathcal{M}, w \Vdash \langle t?\rangle\phi_1$ iff $w \in \mathcal{E}(t)$ and $\mathcal{M}, w \Vdash \phi_1$. Then, by the induction hypothesis, item 6 of definition 11 and the definition of $\mathcal{E}^f(t)$, this happens iff $|w| \in \mathcal{E}^f(t)$ and $\mathcal{M}^f, |w| \Vdash \phi_1$ iff $\mathcal{M}^f, |w| \Vdash \langle t?\rangle\phi_1$.
– If $\psi = t \gg_i \psi_1$, $\psi = \mathcal{P}_i t$ or $\psi = t :_i \psi_1$, the proof is straightforward from the previous cases and items 3, 4 and 5 of the definition 11, respectively. □

Theorem 4 (Finite Model Property). *Every satisfiable formula φ is satisfiable in a finite model.*

Proof. If $\mathcal{M}, w \Vdash \varphi$, we build the set $Cl(\varphi)$ and the finite model $\mathcal{M}^f$, which is the filtration of $\mathcal{M}$ through $Cl(\varphi)$. Then, by the Filtration Theorem, we have $\mathcal{M}^f, |w| \Vdash \varphi$, so φ is satisfiable in a finite model. □

B Strong Completeness

In this section, we present a strong completeness proof for our axiomatization using the standard technique of the construction of *canonical models*, adapting it to the particular features of our logic. For more details on this construction, [10] can be consulted.

Given a consistent set of formulas Δ, we can expand this set using the standard Lindenbaum construction [10] to obtain a maximal consistent set (MCS) $\Delta^+ \supseteq \Delta$. A set Γ is a MCS if it is consistent and all sets $\Gamma' \supsetneq \Gamma$ are inconsistent. Any MCS Γ has the following important property: for every formula φ in the language, $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$.

Definition 13 (Canonical Pre-Model). *The canonical LPJ pre-model is the tuple $\mathcal{M}^C = (W^C, \{\sim_i^C\}_{i\in\mathcal{A}}, \{\geq_i^C\}_{i\in\mathcal{A}}, \mathbf{V}^C, \mathcal{E}^C)$, where:*

– *W^C is the set of all MCS's.*

- *If $\Gamma, \Gamma' \in W^C$, then $\Gamma \sim_i^C \Gamma'$ iff, for all formulas $\psi \in \Gamma'$, $\Gamma \cup \{\langle \mathcal{K}_i \rangle \psi\}$ is consistent (or, in an equivalent definition, $\langle \mathcal{K}_i \rangle \psi \in \Gamma$, since Γ is a MCS).*
- *If $\Gamma, \Gamma' \in W^C$, then $\Gamma \geq_i^C \Gamma'$ iff, for all formulas $\psi \in \Gamma'$, $\Gamma \cup \{\Diamond_i \psi\}$ is consistent ($\Diamond_i \psi \in \Gamma$).*
- $\mathbf{V}^C(p) = \{\Gamma \in W^C : p \in \Gamma\}$, *for all* $p \in \Phi$.
- $\mathcal{E}^C(t) = \{\Gamma \in W^C : \langle t? \rangle \top \in \Gamma\}$, *for all* $t \in \mathcal{T}$.

Lemma 1 (Sound Evidence Lemma). *The canonical evidence function $\mathcal{E}^C$ as defined above satisfies all of the properties required of an evidence function in an LPJ model (definition 9).*

Proof.

- $\Gamma \in \mathcal{E}^C(p)$ iff $\langle p? \rangle \top \in \Gamma$ iff, by axiom $\mathbf{E}_2$, $p \in \Gamma$ iff $\Gamma \in \mathbf{V}^C(p)$, so $\mathcal{E}^C(p) = \mathbf{V}^C(p)$, for all $p \in \Phi$.
- $\Gamma \in \mathcal{E}^C(g)$ iff $\langle g? \rangle \top \in \Gamma$, which is always the case by axiom $\mathbf{E}_3$, so $\mathcal{E}^C(g) = W^C$.
- $\Gamma \in \mathcal{E}^C(t_1 + t_2)$ iff $\langle (t_1 + t_2)? \rangle \top \in \Gamma$ iff, by axiom $\mathbf{E}_5$, $\langle t_1? \rangle \top \in \Gamma$ and $\langle t_2? \rangle \top \in \Gamma$ iff $\Gamma \in \mathcal{E}^C(t_1)$ and $\Gamma \in \mathcal{E}^C(t_2)$, so $\mathcal{E}^C(t_1 + t_2) = \mathcal{E}^C(t_1) \cap \mathcal{E}^C(t_2)$.
- $\Gamma \in \mathcal{E}^C(\bar{t})$ iff $\langle \bar{t}? \rangle \top \in \Gamma$ iff, by axiom $\mathbf{E}_4$, $\neg \langle t? \rangle \top \in \Gamma$ iff $\langle t? \rangle \top \notin \Gamma$ iff $\Gamma \notin \mathcal{E}^C(t)$, so $\mathcal{E}^C(\bar{t}) = W^C \setminus \mathcal{E}^C(t)$. □

It is straightforward to check that $\mathcal{M}^C$ satisfies many of the properties of an LPJ model: $\sim_i^C$ is an equivalence relation (by axioms $\mathbf{T}_{\mathcal{K}}$, $\mathbf{4}_{\mathcal{K}}$ and $\mathbf{5}_{\mathcal{K}}$), $\geq_i^C$ is reflexive (by axiom $\mathbf{T}_{\Box}$) and transitive (by axiom $\mathbf{4}_{\Box}$), we have $\sim_i^C = \geq_i^C \cup (\geq_i^C)^{-1}$ (by axioms $\mathbf{Rel}_1$ and $\mathbf{Rel}_2$) and the evidence function satisfies the desired inductive rules (Lemma 1). Nevertheless, $\mathcal{M}^C$ is not weakly connected. That is why we call $\mathcal{M}^C$ the canonical *pre-model*. So, in order to build an LPJ model useful for proving that the consistent set Δ is satisfiable, we take the MCS Δ^+ and build the set W^{Δ^+} as the maximal weakly connected subset of W^C that contains Δ^+. We then define $\sim_i^{\Delta^+}$, $\geq_i^{\Delta^+}$, $\mathbf{V}^{\Delta^+}$ and $\mathcal{E}^{\Delta^+}$ as the restrictions of $\sim_i^C$, $\geq_i^C$, $\mathbf{V}^C$ and $\mathcal{E}^C$ to the states in W^{Δ^+}, respectively. Then, it is straightforward to verify that $\mathcal{M}^{\Delta^+} = (W^{\Delta^+}, \{\sim_i^{\Delta^+}\}_{i \in \mathcal{A}}, \{\geq_i^{\Delta^+}\}_{i \in \mathcal{A}}, \mathbf{V}^{\Delta^+}, \mathcal{E}^{\Delta^+})$ satisfies all of the above properties that $\mathcal{M}^C$ satisfies and it is also weakly connected by construction. Thus, $\mathcal{M}^{\Delta^+}$ is an LPJ model, and we call it the *canonical model* for the consistent set Δ.

Lemma 2 (Existence Lemma). *Let Γ be a MCS in $\mathcal{M}^{\Delta^+}$. If the formula $\langle \mathcal{K}_i \rangle \psi$ belongs to Γ, then there is a MCS Γ' such that $\Gamma \sim_i^{\Delta^+} \Gamma'$ and $\psi \in \Gamma'$. Analogously, if the formula $\Diamond_i \psi$ belongs to Γ, then there is a MCS Γ'' such that $\Gamma \geq_i^{\Delta^+} \Gamma''$ and $\psi \in \Gamma''$.*

Proof. We show the proof for the $\langle \mathcal{K}_i \rangle$ modality. The proof for the modality $\Diamond_i$ is entirely analogous. Suppose that $\langle \mathcal{K}_i \rangle \psi \in \Gamma$. We want to build a MCS Γ' such that $\psi \in \Gamma'$ and $\Gamma \sim_i^{\Delta^+} \Gamma'$. If $\neg \langle \mathcal{K}_i \rangle \varphi \in \Gamma$ and $\varphi \in \Gamma'$, then we would not have $\Gamma \sim_i^{\Delta^+} \Gamma'$, since Γ is a MCS. So, for all formulas $\neg \langle \mathcal{K}_i \rangle \varphi \in \Gamma$, we must have $\neg \varphi \in \Gamma'$. We can use this fact to build the set $\Sigma = \{\psi\} \cup \{\neg \varphi : \neg \langle \mathcal{K}_i \rangle \varphi \in \Gamma\}$.

If Σ were not consistent, we would have $\{\neg\varphi_1, \ldots, \neg\varphi_k\} \subset (\Sigma \setminus \{\psi\})$, where $\vdash \neg(\psi \wedge \neg\varphi_1 \wedge \ldots \wedge \neg\varphi_k)$. Using propositional reasoning, we get $\vdash \psi \rightarrow (\varphi_1 \vee \ldots \vee \varphi_k)$. Using axioms $\mathbf{K}_{\mathcal{K}}$ and $\mathbf{Du}_{\mathcal{K}}$, rules **MP** and **Gen** and propositional reasoning, we get $\vdash \langle \mathcal{K}_i \rangle \psi \rightarrow (\langle \mathcal{K}_i \rangle \varphi_1 \vee \ldots \vee \langle \mathcal{K}_i \rangle \varphi_k)$, which contradicts the consistency of Γ. Then, Σ is consistent and can be expanded to a MCS Σ^+. We can then take $\Gamma' = \Sigma^+$. □

Lemma 3 (Truth Lemma). *Let $\mathcal{M}^{\Delta^+}$ be the canonical model for the set Δ^+. For all MCS's Γ in the model $\mathcal{M}^{\Delta^+}$ and all formulas φ in the language, $\mathcal{M}^{\Delta^+}, \Gamma \Vdash \varphi$ iff $\varphi \in \Gamma$.*

Proof. The proof is by induction on the structure of the formula φ. For the rest of this proof, Γ will denote a MCS in $\mathcal{M}^{\Delta^+}$.

- If $\varphi = p \in \Phi$, $\varphi = \top$, $\varphi = \neg\psi$ or $\varphi = \varphi_1 \wedge \varphi_2$, the proof is straightforward from the definitions of $\mathbf{V}^C$ and $\mathbf{V}^{\Delta^+}$ and from the definition of a MCS.
- If $\varphi = \mathcal{K}_i \psi$, then, by the induction step for negation and axiom $\mathbf{Du}_{\mathcal{K}}$, it is sufficient to show the result for formulas of the form $\phi = \langle \mathcal{K}_i \rangle \phi_1$.
  ($\Rightarrow$) Suppose that $\langle \mathcal{K}_i \rangle \phi_1 \in \Gamma$. Then, by the Existence Lemma, there is Γ' in $\mathcal{M}^{\Delta^+}$ such that $\Gamma \sim_i^{\Delta^+} \Gamma'$ and $\phi_1 \in \Gamma'$. By the induction hypothesis, $\mathcal{M}^{\Delta^+}, \Gamma' \Vdash \phi_1$, which implies $\mathcal{M}^{\Delta^+}, \Gamma \Vdash \langle \mathcal{K}_i \rangle \phi_1$.
  ($\Leftarrow$) Suppose that $\mathcal{M}^{\Delta^+}, \Gamma \Vdash \langle \mathcal{K}_i \rangle \phi_1$. Then, there is Γ' in $\mathcal{M}^{\Delta^+}$ such that $\Gamma \sim_i^{\Delta^+} \Gamma'$ and $\mathcal{M}^{\Delta^+}, \Gamma' \Vdash \phi_1$. By the induction hypothesis, $\phi_1 \in \Gamma'$ and, by the definitions of $\sim_i^C$ and $\sim_i^{\Delta^+}$, we must have $\langle \mathcal{K}_i \rangle \phi_1 \in \Gamma$.
- If $\varphi = \Box_i \psi$, then the proof is entirely analogous to the previous case, using now axiom $\mathbf{Du}_{\Box}$ and the second part of the Existence Lemma.
- If $\varphi = [t?]\psi$, then, using the same reasoning of the previous two cases and axiom $\mathbf{Du}_t$, it is sufficient to show the result for formulas of the form $\phi = \langle t? \rangle \phi_1$. By axiom $\mathbf{E}_1$, $\langle t? \rangle \phi_1 \in \Gamma$ iff $\langle t? \rangle \top \in \Gamma$ and $\phi_1 \in \Gamma$. Now, by the definitions of $\mathcal{E}^C$ and $\mathcal{E}^{\Delta^+}$, $\langle t? \rangle \top \in \Gamma$ iff $\Gamma \in \mathcal{E}^{\Delta^+}(t)$ and, by the induction hypothesis, $\phi_1 \in \Gamma$ iff $\mathcal{M}^{\Delta^+}, \Gamma \Vdash \phi_1$. Finally, $\Gamma \in \mathcal{E}^{\Delta^+}(t)$ and $\mathcal{M}^{\Delta^+}, \Gamma \Vdash \phi_1$ iff $\mathcal{M}^{\Delta^+}, \Gamma \Vdash \langle t? \rangle \phi_1$.
- If $\varphi = t \gg_i \psi$, $\varphi = \mathcal{P}_i t$ or $\varphi = t :_i \psi_1$, the proof is straightforward from the previous cases and axioms **Adm**, **Pla** and **Jus**, respectively. □

Theorem 5 (Completeness). *Every consistent set of formulas is satisfiable in an LPJ model (definition 9).*

Proof. Let Δ be a consistent set of formulas and $\Delta^+ \supseteq \Delta$ be a MCS obtained from Δ. Consider the canonical model $\mathcal{M}^{\Delta^+}$. Then, by the Truth Lemma, we conclude that $\mathcal{M}^{\Delta^+}, \Delta^+ \Vdash \varphi$, for all $\varphi \in \Delta$, so Δ is satisfiable. □