Hybrid Logics and NP Graph Properties

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Abstract

We show that for each property of graphs $\mathcal{G}$ in NP there is a sequence $\phi_1, \phi_2, \ldots$ of formulas of the full hybrid logic which are satisfied exactly by the frames in $\mathcal{G}$. Moreover, the size of ϕ_n is bounded by a polynomial. We also show that the same holds for each graph property in the polynomial hierarchy. These results lead to the definition of syntactically defined fragments of hybrid logic whose model checking problem is complete for each degree in the polynomial hierarchy.

1 Introduction

The use of graphs as a mathematical abstraction of objects and structures makes it one of the most used concepts in computer science. Several typical problems in computer science have their inputs modeled by graphs, and such problems commonly involve evaluating some graph property. To mention a well-known example, deciding whether a map can be colored with a certain number of colors is related to a similar problem on planar graphs [HAK89, RSST97]. The applications of graphs in computer science are not restricted to modelling the input of problems. Graphs can be used in the theoretical framework in which some branches of computer science are formalized. This is the case, for example, in distributed systems, in which the model of computation is built on top of a graph [Bar96, Lyn96]. Again, properties of graphs can be exploited in order to obtain results about such models of computation.

We can use logic to express properties of structures like graphs. From the semantical standpoint, a logic can be regarded as a pair $\mathcal{L} = (L, \models)$, where L, the language of $\mathcal{L}$, is a set of elements called formulas, and $\models$ is a binary satisfaction relation between some set of objects or structures and formulas. Such set of structures can be a set of relational structures, for example, the class of graphs. A sentence ϕ of L can be used to express some property of structures, hence one could check whether a structure $\mathfrak{A}$ has some property by evaluating whether $\mathfrak{A} \models \phi$ holds or not. The problem of checking whether a given model satisfies a given formula is called model checking.

In the last few decades, modal logics have attracted the attention of computer scientists working with logic and computation [BDRV02]. Among the reasons is the fact that modal logics often have interesting computer theoretical properties, like decidability [Var96, Grä01]. This is due to a lack of expressive power in comparison with other logics such as first-order logics and its extensions. Many modal logics present also good logical properties, like interpolation, definability, and so on. Research in modal logic includes augmenting the expressive power of the logic using resources as fixed-point operators [BS07] or hybrid languages [AtC07, ABM01]. Modal logics are particularly suitable to deal with graphs because the models of most modal logics are built up from structures called *frames*, which are essentially graphs.

In [BS09], hybrid logics are used to express graph properties, like being connected, hamiltonian or eulerian. Several hybrid logics and fragments were studied to define graph properties through the concept of *validity in a frame* (see Definition 5 below). Some graph properties, like being hamiltonian, require a high expressive power and cannot be expressed by a single sentence in traditional hybrid logics. There are, however, sentences ϕ_n which can express such properties for frames of size n.

We are interested in expressing graph properties in NP using hybrid logics. Hybrid Modal logics have low expressive power, hence we do not aim to associate to each graph property a single formula. Instead, we present, to each graph property, a sequence of hybrid sentences $\phi_1, \phi_2, \ldots$, such that a graph of size n has the desired property iff ϕ_n is valid in the graph, regarded as a frame. In Section 2, we define the hybrid logic which we will study and define a prenex form for such logic. In Section 3, we show that, for any graph property, there is a sequence of sentences $\phi_1, \phi_2, \ldots$ of the fragment of hybrid logic with nominals and the @ operator such that a graph of size n has the property iff ϕ_n is valid in the corresponding frame. However, the size of ϕ_n obtained is exponential on n. In Section 4, we show that, for graph properties in NP, and more generally in the polynomial hierarchy, there is such a sequence, but the size of the sentences is bounded by a polynomial on n. In Section 5, we show how to obtain the results of the previous section for the fragment of hybrid logic without the global modality E and without nominals, provided that graphs are connected. In Section 6, we show fragments of hybrid logic whose model-checking problem is complete for each degree of the polynomial hierarchy based on the results of the other sections. This gives an alternative proof for the NP-hardness of the model-checking problems for the fragment FHL$\setminus \downarrow \Box \downarrow$ of full hybrid logic given in [tCF05]. The proofs of the main theorems appears in Appendix A.

2 Hybrid Logic

In this section, we present the hybrid logic and its fragments which we will use. Hybrid modal logics extend classical modal logics by adding nominals and state variables to the language. Nominals and state variables behave like propositional atoms which are true in exactly one world. Other extensions include the operators $\downarrow$ (binder) and @. The $\downarrow$ allows one to assign the current state to a state variable. This can be used to keep a record of the visited states. The @ operator allows one to evaluate a formula in the state assigned to a certain nominal or state variable.

Definition 1. *The language of the* hybrid graph logic with the $\downarrow$ binder *is a hybrid language consisting of a set $PROP$ of countably many proposition symbols $p_1, p_2, \ldots$, a set NOM of countably many nominals $i_1, i_2, \ldots$, a set $\mathcal{S}$ of countably many state-variables $x_1, x_2, \ldots$, such that $PROP$, NOM and $\mathcal{S}$ are pairwise disjoint, the boolean connectives $\neg$ and $\wedge$ and the modal operators $@_i$, for each nominal i, $@_x$, for each state-variable x, $\Diamond$, $\Diamond^{-1}$ and $\downarrow$. The language L_{FHL} of the (Full) Hybrid Logic can be defined by the following BNF rule:*

$$\alpha := p \mid t \mid \neg\alpha \mid \alpha \wedge \alpha \mid \Diamond\alpha \mid \Diamond^{-1}\alpha \mid E\alpha \mid @_t\alpha \mid \ \downarrow x.\alpha \mid \top,$$

where t is either a nominal or a state variable. For each $C \subseteq \{@, \downarrow, \Diamond^{-1}, E\}$, we define $HL(C)$ to be the corresponding fragment. In particular, we define $FHL = HL(@, \downarrow, \Diamond^{-1}, E)$. We also use $HL(C) \backslash NOM$ and $HL(C) \backslash PROP$ to refer to the fragments of $HL(C)$ without nominals and propositional symbols respectively.

The standard boolean abbreviations $\rightarrow$, $\leftrightarrow$, $\vee$ and $\bot$ can be used with the standard meaning as well as the abbreviations of the dual modal operators: $\Box\phi := \neg\Diamond\neg\phi$, $\Box^{-1}\phi := \neg\Diamond^{-1}\neg\phi$ and $A\phi := \neg E \neg\phi$.

Formulas of hybrid modal logics are evaluated in *hybrid Kripke structures* (or *hybrid models*). These structures are build from *frames*.

Definition 2. *A* frame *is a graph $\mathcal{F} = (W, R)$, where W is a non-empty set (finite or not) of vertices and R is a binary relation over W, i.e., $R \subseteq W \times W$.*

Definition 3. *A* (hybrid) model *for the hybrid logic is a pair $\mathcal{M} = (\mathcal{F}, \mathbf{V})$, where $\mathcal{F}$ is a frame and $\mathbf{V} : PROP \cup NOM \mapsto \mathcal{P}(W)$ is a valuation function mapping proposition symbols into subsets of W, and mapping nominals into singleton subsets of W, i.e, if i is a nominal then $\mathbf{V}(i) = \{v\}$ for some $v \in W$.*

In order to deal with the state-variables, we need to introduce the notion of *assignments*.

Definition 4. *An* assignment *is a function g that maps state-variables to vertices of the model, $g : \mathcal{S} \mapsto W$. We use the notation $g' = g[v_1/x_1, \ldots, v_n/x_n]$ to denote an assignment such that $g'(x) = g(x)$ if $x \notin \{x_1, \ldots, x_n\}$ and $g'(x_i) = v_i$, otherwise.*

The semantical notion of satisfaction is defined as follows:

Definition 5. *Let $\mathcal{M} = (\mathcal{F}, \mathbf{V})$ be a model. The notion of* satisfaction *of a formula φ in a model $\mathcal{M}$ at a vertex v with assignment g, notation $\mathcal{M}, g, v \Vdash \varphi$, can be inductively defined as follows:*

$\mathcal{M}, g, v \Vdash p$ *iff* $v \in \mathbf{V}(p)$;
$\mathcal{M}, g, v \Vdash \top$ *always;*
$\mathcal{M}, g, v \Vdash \neg\varphi$ *iff* $\mathcal{M}, g, v \nVdash \varphi$;
$\mathcal{M}, g, v \Vdash \varphi_1 \wedge \varphi_2$ *iff* $\mathcal{M}, g, v \Vdash \varphi_1$ *and* $\mathcal{M}, g, v \Vdash \varphi_2$;
$\mathcal{M}, g, v \Vdash \Diamond\varphi$ *iff there is a* $w \in W$ *such that* vRw *and* $\mathcal{M}, g, w \Vdash \varphi$;
$\mathcal{M}, g, v \Vdash \Diamond^{-1}\varphi$ *iff there is a* $w \in W$ *such that* wRv *and* $\mathcal{M}, g, w \Vdash \varphi$;
$\mathcal{M}, g, v \Vdash i$ *iff* $v \in \mathbf{V}(i)$;
$\mathcal{M}, g, v \Vdash @_i\varphi$ *iff* $\mathcal{M}, g, d_i \Vdash \varphi$, *where* $d_i \in \mathbf{V}(i)$;
$\mathcal{M}, g, v \Vdash x$ *iff* $g(x) = v$;
$\mathcal{M}, g, v \Vdash @_x\phi$ *iff* $\mathcal{M}, g, d \Vdash \phi$, *where* $d = g(x)$;
$\mathcal{M}, g, v \Vdash \downarrow x.\phi$ *iff* $\mathcal{M}, g[v/x], v \Vdash \phi$.

For each nominal i, the formula $@_i\varphi$ means that if $\mathbf{V}(i) = \{v\}$ then φ is satisfied at v. If $\mathcal{M}, g, v \Vdash \varphi$ for every vertex v, we say that φ is *globally satisfied* in the model $\mathcal{M}$ with assignment g ($\mathcal{M}, g \Vdash \varphi$) and if φ is globally satisfied in all models $\mathcal{M}$ and assignments of a frame $\mathcal{F}$, we say that φ is *valid* in $\mathcal{F}$ ($\mathcal{F} \Vdash \varphi$). The next lemma follows directly from the definition of satisfaction.

Lemma 1. *The following equivalences hold in HL:*

- $(\Box\alpha \wedge \beta) \equiv \downarrow x.\Box(\alpha \wedge @_x\beta)$, $(A\alpha \wedge \beta) \equiv \downarrow x.A(\alpha \wedge @_x\beta)$;
- $(\Box\alpha \vee \beta) \equiv \downarrow x.\Box(\alpha \vee @_x\beta)$, $(A\alpha \vee \beta) \equiv \downarrow x.A(\alpha \vee @_x\beta)$;
- $(\Diamond\alpha \wedge \beta) \equiv \downarrow x.\Diamond(\alpha \wedge @_x\beta)$, $(E\alpha \wedge \beta) \equiv \downarrow x.E(\alpha \wedge @_x\beta)$;
- $(\Diamond\alpha \vee \beta) \equiv \downarrow x.\Diamond(\alpha \vee @_x\beta)$, $(E\alpha \vee \beta) \equiv \downarrow x.E(\alpha \vee @_x\beta)$;
- $(\Diamond^{-1}\alpha \wedge \beta) \equiv \downarrow x.\Diamond^{-1}(\alpha \wedge @_x\beta)$, $(\Diamond^{-1}\alpha \vee \beta) \equiv \downarrow x.\Diamond^{-1}(\alpha \vee @_x\beta)$;
- $((\downarrow x.\alpha) \wedge \beta) \equiv \downarrow x.(\alpha \wedge \beta)$, $((\downarrow x.\alpha) \vee \beta) \equiv \downarrow x.(\alpha \vee \beta)$.

In the following, we define a prenex form for formulas in FHL and show that any formula in FHL has an equivalent in prenex form. We use this form to define classes of formulas whose model-checking problem is complete for the degrees of the polynomial hierarchy (see Section 6).

Definition 6 (Prenex Form)**.** *A formula ϕ in FHL is in* prenex form *iff $\phi = q_1 \dots q_n\psi$ where each q_i is $\Box$, $\Box^{-1}$, $\Diamond$, $\Diamond^{-1}$, E, A or $\downarrow x.$, for some x, and ψ has no ocurrence of modalities or $\downarrow$.*

It follows from Lemma 1 that each formula of FHL can be put in this prenex form.

Lemma 2. *If $\phi \in$ FHL, then there is $\psi \in$ FHL in prenex form which is equivalent to ϕ.*

This prenex form can be strengthened with the following lemma:

Lemma 3. $\downarrow x. \downarrow y.\phi(x, y) \equiv \downarrow x.\phi(x, x)$, *if x does not occur bound in ϕ.*

Lemma 4. *If $\phi \in FHL$, then there is $\psi \in FHL$ without modalities or binders and a prefix $\overline{q} = q_1 \dots q_n$ where each q_i is $\Box$, $\Box^{-1}$, $\Diamond$, $\Diamond^{-1}$, E, A or $\downarrow x.$ for some x, and there is no consecutive application of binders in $\overline{q}$.*

Based on Lemma 4, we define the following classes of formulas:

Definition 7. *Let* $\mathrm{EX} = \{\Diamond, \Diamond^{-1}, E\}$ *and* $\mathrm{UN} = \{\Box, \Box^{-1}, A\}$. *We recursively define the classes of formulas σ^i and π^i in prenex form as:*

- $\sigma^0 = \pi^0 = \{\phi \in HL | \phi \text{ has no modalities}\}$;
- $\sigma^{i+1} = \{\phi \in HL | \phi = q_1 \dots q_n \psi, \psi \in \pi^i, q_j \in \mathrm{EX} \cup \{\downarrow x.\}, \text{ for some } x\}$;
- $\pi^{i+1} = \{\phi \in HL | \phi = q_1 \dots q_n \psi, \psi \in \sigma^i, q_j \in \mathrm{UN} \cup \{\downarrow x.\}, \text{ for some } x\}$.

We say that a formula is σ^i (resp. π^i) if it is equivalent to a formula in σ^i (resp. π^i).

From Lemma 2 it follows that each formula in HL is π^i or σ^i for some i.

3 Properties of Graphs in HL

In [BS09], it was shown that there is a formula ϕ_n of FHL such that a graph of size n is Hamiltonian iff it globally satisfies ϕ_n. The main question which underlies this investigation is whether there is a sequence of formulas $(\phi_n)_{n \in N}$ for each graph property $\mathcal{G}$ in NP such that a graph G of size n is in $\mathcal{G}$ iff G, as a frame, globally satisfies ϕ_n. Actually, we can show that such sequence exists for each graph property.

Let $G = (V, E)$ be a graph of cardinality n. Let us consider that the set V of vertices coincides with the set $\{1, \dots, n\}$ of nominals. Consider the formula:

$$\psi_G = \bigwedge_{(i,j) \in E} @_i \Diamond j \wedge \bigwedge_{(i,j) \notin E} @_i \neg \Diamond j.$$

Let $\mathcal{G}$ be any property of graphs. We define the formulas

$$\psi^n_{\mathcal{G}} = \bigvee_{G \in \mathcal{G}, |G| = n} \psi_G \quad , \quad \theta^n = \bigwedge_{i,j \in \{1,\dots,n\}, i \neq j} @_i \neg j \quad \text{and} \quad \phi^n_{\mathcal{G}} = \theta^n \to \psi^n_{\mathcal{G}}.$$

Lemma 5. *Let G be a graph of cardinality n and $\mathcal{G}$ a property of graphs. Then $G \in \mathcal{G}$ iff $G \Vdash \phi^n_{\mathcal{G}}$.*

Since there are 2^{n^2} graphs with vertices in $\{1, \dots, n\}$, we have that the size of $\phi^n_{\mathcal{G}}$ is $O(2^{n^2})$ for any graph property $\mathcal{G}$. Obviously, there is no hope for that sequence of formulas to be always computable. We can show, however, that, for problems in the polynomial hierarchy, such sequence is recursive and, moreover, there is a polynomial bound in the size of formulas.

4 Translation

In this section, we show that for each graph property $\mathcal{G}$ in the polynomial hierarchy there is a sequence $(\phi_n)_{n\in N}$ of formulas such that a graph G of size n is in $\mathcal{G}$ iff $G \Vdash \phi_n$ and such that ϕ_n is bounded from above by a polynomial on n. We will use the well-known characterization of problems in PH and classes of finite models definable in second-order logic (SO) from descriptive complexity theory [Imm99]. To this end, we define a translation from formulas in SO to formulas in FHL which are equivalent with respect to frames of size n, for some $n \in N$. Such translation will give us formulas whose size is bounded by a polynomial on n. Moreover, the formulas obtained by the translation do not use propositional symbols, nominals or free state variables, which means that, for these formulas, the complexity of model-checking and frame-checking coincides. We use the well known definitions and concepts related to first-order logic (FO) and second-order logic which can be founded in most textbooks (see, for instance, [EFT94]).

Definition 8 (Translation from FO to HL)**.** *Let ϕ be a first-order formula in the vocabulary $S = \{E, R_1, \ldots, R_m\}$ where E is binary, n a natural number and f a function from the set of first-order variables into $\{1, \ldots, n\}$. Let $t, z_1, \ldots, z_n$ be state variables and for each $R \in \{R_1, \ldots, R_m\}$ of arity h, let $y^R_{j_1,\ldots,j_h}$ be a state variable, with $j_i \in \{1, \ldots, n\}$, $1 \leq i \leq h$. We define the function $tr^f_n : L^S_{FO} \to L_{FHL}$ as:*

- $tr^f_n(x_1 \equiv x_2) = @_{z_{f(x_1)}} z_{f(x_2)}$;
- $tr^f_n(E(x_1, x_2)) = @_{z_{f(x_1)}} \Diamond z_{f(x_2)}$;
- $tr^f_n(R(x_1, \ldots, x_k)) = @_t y^R_{f(x_1),\ldots,f(x_k)}$, *for each* $R \in \{R_1, \ldots, R_m\}$;
- $tr^f_n(\gamma \wedge \theta) = tr^f_n(\gamma) \wedge tr^f_n(\theta)$;
- $tr^f_n(\neg\gamma) = \neg tr^f_n(\gamma)$;
- $tr^f_n(\exists x\gamma) = \bigvee^n_{i=1} tr^{f\frac{x}{i}}_n(\gamma)$;
- $tr^f_n(\forall x\gamma) = \bigwedge^n_{i=1} tr^{f\frac{x}{i}}_n(\gamma)$.

In the translation above, t is intended to represent a state v such that, if $z^R_{j_1,\ldots,j_h}$ is assigned to v and $z_{j_1}, \ldots, z_{j_h}$ are assigned to $v_1, \ldots, v_h$, then $(v_1, \ldots, v_h)$ belongs to the interpretation of R.

Note that if ϕ is a sentence, then $tr^f_n(\phi) = tr^{f'}_n(\phi)$. Hence we write $tr_n(\phi)$ instead of $tr^f_n(\phi)$ for a sentence ϕ.

Example 1. We give an example of application of the translation above. Let $\phi \oplus \psi$ (exclusive "or") be an abbreviation for $(\phi \vee \psi) \wedge \neg(\phi \wedge \psi)$. Consider the following first-order sentence:

$$\begin{aligned} \phi \quad := \quad & \forall x\big(R(x) \oplus G(x) \oplus B(x)\big) \wedge \forall x \forall y\big((E(x,y) \wedge x \neq y) \to \\ & (\neg(R(x) \wedge R(y)) \vee (G(x) \wedge G(y)) \vee (B(x) \wedge B(y)))\big). \end{aligned}$$

The sentence above says that each element belongs to one of the sets R, G and B, each adjacent pair does not belong to the same set, and no element belongs to more than one set. This sentence is true iff the sets R, G and B forms a 3-coloring of a graph with edges in E. Below we translate ϕ into a formula of hybrid logic using the translation given above and setting $n = 3$:

$$\begin{aligned} tr_n(\phi) \quad := \quad & \bigwedge_{i=1}^{3} \left(@_t y_i^R \oplus @_t y_i^G \oplus @_t y_i^B\right) \wedge \bigwedge_{i=1}^{3} \left[\bigwedge_{j=1}^{3} \left((@_{z_i} \Diamond z_j \wedge \neg @_{z_i} z_j\right) \rightarrow \right. \\ & \left. \neg((@_t y_i^R \wedge @_t y_j^R) \vee (@_t y_i^G \wedge @_t y_j^G) \vee (@_t y_i^B \wedge @_t y_j^B))) \right]. \end{aligned}$$

Lemma 6. *$tr_3(\phi)$ has polynomial size in n, that is, $tr_n(\phi) \in O(n^k)$ for some $0 \leq k$.*

Proof. By induction on ϕ one can see that $tr_n(\phi)$ is $O(n^k)$, where k is the quantifier rank of ϕ, that is, the maximum number of nested quantifiers. □

Lemma 7. *Let $G = (V, E^G)$ be a graph of cardinality n, $\mathbf{R}_1, \ldots, \mathbf{R}_m$ relations on V with arities $r_1, \ldots, r_m$, g an assignment of state variables, β an assignment of first-order variables, $S = \{E, R_1, \ldots, R_m\}$ a vocabulary and f a function from the set of first-order variables to $\{1, \ldots, n\}$ such that:*
(i) g assigns to each variable z_i a different element in V;
(ii) $g(y^R_{i_1,\ldots,i_k}) = g(t)$ iff $(g(z_{i_1}), \ldots, g(z_{i_k})) \in \mathbf{R}$ for each $R \in \{R_1, \ldots, R_m\}$;
(iii) $\beta(x) = g(z_{f(x)})$ for each first-order variable x.
If ϕ is a first-order formula in the vocabulary S, then $(G, \mathbf{R}_1, \ldots, \mathbf{R}_m, \beta) \models \phi$ iff for all $w \in V$, $(G, g, w) \Vdash tr_n^f(\phi)$.

Definition 9 (Translation from SO to FHL)**.** *Let $\phi = Q_1 X_1 \ldots Q_l X_l \psi$ be a SO formula where $Q_i \in \{\exists, \forall\}$ and ψ is a first-order sentence. We define*

$$T^n(\phi) = \spadesuit_1 \downarrow y_{\overline{1}}^{X_1}.\ldots.\spadesuit_1 \downarrow y_{\overline{n}}^{X_1}.\ldots.\spadesuit_l \downarrow y_{\overline{1}}^{X_l}.\ldots.\spadesuit_l \downarrow y_{\overline{n}}^{X_l} tr_n(\psi),$$

where $\spadesuit_i = E$ if $Q_i = \exists$ and A otherwise.

Example 2. Consider the sentence ϕ of Example 1. Let ψ be the following second-order sentence:

$$\psi := \exists R \exists G \exists B(\phi).$$

The sentence ψ above states that there are three sets R, G and B which forms a 3-coloring of elements in the domain of a structure. Hence, ϕ is satisfied in a graph with edges in E iff such graph is 3-colorable. Deciding whether a graph is 3-colorable is a NP-complete problem [Pap03]. We apply the translation T^n for $n = 3$ below. Let

$$\hat{\mathbf{Q}} := E \downarrow y_1^R.E \downarrow y_2^R.E \downarrow y_3^R.E \downarrow y_1^G.E \downarrow y_2^G.E \downarrow y_3^G.E \downarrow y_1^B.E \downarrow y_2^B.E \downarrow y_3^B..$$

We have $T^3(\psi) := \hat{\mathbf{Q}} tr_3(\phi)$. That is,

$$\begin{aligned} T^3(\phi) \quad := \quad & \hat{\mathbf{Q}} \left(\bigwedge_{i=1}^{3} \left(@_t y_i^R \oplus @_t y_i^G \oplus @_t y_i^B\right) \wedge \bigwedge_{i=1}^{3} \left[\bigwedge_{j=1}^{3} \left((@_{z_i} \Diamond z_j \wedge \neg @_{z_i} z_j\right) \rightarrow \right.\right. \\ & \left.\left. \neg((@_t y_i^R \wedge @_t y_j^R) \vee (@_t y_i^G \wedge @_t y_j^G) \vee (@_t y_i^B \wedge @_t y_j^B))) \right] \right). \end{aligned}$$

Lemma 8. *Let $G = (V, E^G)$ be a graph of cardinality n, $\mathbf{R}_1, \ldots, \mathbf{R}_m$ relations on V with arities $r_1, \ldots, r_m$, g an assignment of state variables, β an assignment of first-order variables, $S = \{E, R_1, \ldots, R_m\}$ a vocabulary and f a function from the set of first-order variables to $\{1, \ldots, n\}$ such that:*

(i) g assigns to each variable z_i a different element in V;

(ii) $g(y^R_{i_1,\ldots,i_k}) = g(t)$ iff $(g(z_{i_1}), \ldots, g(z_{i_k})) \in \mathbf{R}$ for each $R \in \{R_1, \ldots, R_m\}$;

(iii) $\beta(x) = g(z_{f(x)})$ for each first-order variable x.

If $\phi = Q_1X_1 \ldots Q_lX_l\psi$ is a second-order formula in the symbol set S, then $(G, \mathbf{R}_1, \ldots, \mathbf{R}_m, \beta) \models \phi$ iff for all $w \in V$, $(G, g, w) \Vdash T^n(\phi)$.

We have the following:

Theorem 1. *Let ϕ be a second-order sentence and G a graph of cardinality n. Then $G \models \phi$ iff*

$$G \Vdash \downarrow t.E \downarrow z_1.\ldots E \downarrow z_n.\left(\bigwedge_{1\leq i<j\leq n} @_{z_i}\neg z_j \wedge T^n(\phi)\right).$$

A well known result of descriptive complexity is the correspondence between the polynomial hierarchy and the alternation hierarchy of second-order logic (with respect to finite models) [Imm99, Fag74].

There are several ways to define this hierarchy, for example using alternating Turing machines [Imm99]. In this paper we assume the definition presented in [Pap03], which uses Turing machines with oracles to define PH.

A Turing machine with an oracle is a machine that has the special ability of guessing some specific questions. When a Turing machine has an oracle for a decision problem B, during its execution it can ask for the oracle if some instance of problem B is positive or negative. This is done in constant time, regardless the size of the instance. We use the notation M^B to define a Turing machine M with an oracle for a problem B. In a similar way, we define $\mathcal{C}^{\mathcal{B}}$, where $\mathcal{C}$ and $\mathcal{B}$ are complexity classes, as the class of problems solved by a Turing machine in $\mathcal{C}$ with an oracle in $\mathcal{B}$.

Definition 10. *Consider the following sequence of complexity classes. First, $\Delta^p_0 = \Sigma^p_0 = \Pi^p_0 = PTIME$ and, for all $i \geq 0$,*

1. $\Delta^p_{i+1} = P^{\Sigma^p_i}$
2. $\Sigma^p_{i+1} = NP^{\Sigma^p_i}$
3. $\Pi^p_{i+1} = coNP^{\Sigma^p_i}$.

We define the Polynomial Time Hierarchy as the class $PH = \bigcup_{i\geq 0} \Sigma^p_i$.

In particular we have:

Theorem 2 ([Imm99])**.** *Let $\mathcal{G}$ be a graph property in the polynomial hierarchy. Then there is a second-order sentence ϕ in the language of graphs such that $G \in \mathcal{G}$ iff $G \models \phi$.*

The following is the main theorem of this section:

Theorem 3. *Let $\mathcal{G}$ be a graph property in the polynomial hierarchy. Then there is a set of sentences $\Phi = \{\phi_1, \phi_2, \ldots\}$ of FHL, such that:*

(1) $G \in \mathcal{G}$ iff $G \Vdash \Phi$ iff $G \Vdash \phi_{|G|}$, and

(2) ϕ_m is $O(n^k)$ for some constant k depending only on $\mathcal{G}$.

Corollary 1. *If $\phi \in \exists SO$, the existencial fragment of SO, then T^n is in HL$(@, \downarrow, E)\backslash\{\downarrow \Box \downarrow, PROP\}$, that is, the fragment of HL$(@, \downarrow, E)$ without propositional symbols and the pattern $\downarrow \Box \downarrow$.*

5 Connected Frames with Loops

Let FHL$\backslash\{E, NOM\}$ be the fragment of full hybrid logic without the modality E and without nominals. Its is not difficult to show that:

Lemma 9. *Frame validity and model (global) satisfaction for sentences from FHL$\backslash\{E, NOM\}$ are invariant under disjoint union.*

Thus an analogous to Theorem 3 does not hold for FHL$\backslash\{E, NOM\}$.

Corollary 2. *There are graph properties in PTIME for which there is no set $\Phi = \{\phi_1, \phi 2, \ldots\}$ from sentences in FHL$\backslash\{E, NOM\}$ which satisfies conditions (1) and (2) from Theorem 3 above.*

Proof. Connectivity is one such a property. □

However, Theorem 3 still hold if we restrict ourselves to connected, frames with loops. Consider the following translation from SO to FHL$\backslash\{E, NOM\}$:

Definition 11. *Let $\phi = Q_1 X_1 \ldots Q_l X_l \psi$ be a second-order formula where $Q_i \in \{\exists, \forall\}$ and ψ is a first-order sentence. We define*

$$\hat{T}^n(\phi) = \spadesuit_1 \downarrow y^{X_1}_{\overline{1}} \ldots . \spadesuit_1 \downarrow y^{X_1}_{\overline{n}} \ldots . \spadesuit_l \downarrow y^{X_l}_{\overline{1}} \ldots . \spadesuit_l \downarrow y^{X_l}_{\overline{n}} tr_n(\psi),$$

where $\spadesuit_i = (\Diamond\Diamond^{-1})^n$ if $Q_i = \exists$ and $(\Box\Box^{-1})^n$ otherwise.

Lemma 10. *Let $G = (V, E^G)$ be a connected graph with loops on each vertex, $\mathbf{R}_1, \ldots, \mathbf{R}_m$ relations on V with arities $r_1, \ldots, r_m$, g an assignment of state variables, β an assignment of first-order variables, $S = \{E, R_1, \ldots, R_m\}$ a vocabulary and f a function from the set of first-order variables to $\{1, \ldots, n\}$ such that:*

(i) g assigns to each variable z_i a different element in V;

(ii) $g(y^R_{i_1,\ldots,i_k}) = g(t)$ iff $(g(z_{i_1}), \ldots, g(z_{i_k})) \in \mathbf{R}$ for each $R \in \{R_1, \ldots, R_m\}$;

(iii) $\beta(x) = g(z_{f(x)})$ for each first-order variable x.

If $\phi = Q_1 X_1 \ldots Q_l X_l \psi$ is a second-order formula in the symbol set S, then

$$(G, \mathbf{R}_1, \ldots, \mathbf{R}_m, \beta) \models \phi \text{ iff for all } w \in V,\ (G, g, w) \Vdash \hat{T}^n(\phi).$$

Proof. Analogous to the proof of Lemma 8 □

Theorem 4. *Let ϕ be a second-order sentence and G a connected graph of cardinality n with loops. Then $G \models \phi$ iff*

$$G \Vdash \downarrow t.(\Diamond\Diamond^{-1})^n \downarrow z_1. \ldots (\Diamond\Diamond^{-1})^n \downarrow z_n. \left(\bigwedge_{1 \leq i < j \leq n} @_{z_i} \neg z_j \wedge \hat{T}^n(\phi) \right).$$

Proof. Analogous to the proof of Theorem 1 □

Theorem 5. *Let $\mathcal{G}$ be a property of connected graphs with loops in the polynomial hierarchy. There is a set of sentences $\Phi = \{\phi_1, \phi_2, \ldots\}$ of the fragment $FHL \backslash \{E, NOM\}$, such that:*

(1)for all connected graphs G with loops, $G \in \mathcal{G}$ iff $G \Vdash \Phi$ iff $G \Vdash \phi_{|G|}$, and

(2) ϕ_m is $O(n^k)$ for some constant k depending only on $\mathcal{G}$.

6 Polynomial Hierarchy

In [tCF05], it is proved that the model checking for the FHL$\backslash \downarrow \Box \downarrow$ fragment is NP-complete. The translation given in Section 4 above can be used to produce hybrid formulas of polynomial size using formulas of second-order logic. This leads to an alternative proof that the model checking problem for the fragment FHL$\backslash \downarrow \Box \downarrow$ is hard for NP, since there is a polynomial reduction for any instance of a NP problem to the model checking of FHL$\backslash \downarrow \Box \downarrow$.

Theorem 6 ([tCF05])**.** *The model checking problem for $\sigma^1 \subseteq FHL \backslash \downarrow \Box \downarrow$ is NP-hard.*

Actually, for each degree of the polynomial hierarchy, there is a syntactically defined fragment of HL whose model checking problem is hard.

Theorem 7. *The model checking problem for σ^i (resp. π^i) is Σ_i^p-hard (resp. Π_i^p-hard).*

Also, the model checking for σ^i and π^i are in Σ_i^p and Π_i^p, respectively.

Theorem 8. *The model checking problem for σ^i (resp. π^i) is in Σ_i^p (resp. Π_i^p).*

Corollary 3. *Let $\Phi = \{\phi_1, \phi_2, \ldots\}$ be such that each ϕ_i can be constructed in time polynomial on i and each ϕ_i is in π^j (resp. σ^j). Then the graph property $\mathcal{G}$ defined as:*

$$G \in \mathcal{G} \text{ iff } G \Vdash \phi_{|G|}$$

is in Π_j^p (resp. Σ_j^p).

From Theorems 7 and 8 we have:

Corollary 4. *The model checking problem for σ^i (resp. π^i) is Σ_i^p-complete (resp. Π^i-complete).*

Corollary 5. *The frame checking problem for sentences in $\sigma^i \backslash \{PROP, NOM\}$ (resp. $\pi^i \backslash \{PROP, NOM\}$) is Σ_i^p-complete (resp. Π^i-complete).*

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A Proofs of Main Theorems

Proof of Lemma 7. We proceed by induction on ϕ.

— ϕ is atomic: In this case $\phi = x \equiv y$, $\phi = E(x,y)$ or $\phi = R_i(x_1,\ldots,x_n)$. If $\phi = x \equiv y$, then $(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta) \models \phi$ iff $\beta(x) = \beta(y)$ iff, by (iii), $g(z_{f(x)}) = g(z_{f(y)})$ iff $(G,g,w) \Vdash @_{z_{f(x)}} z_{f(y)} = tr_n^f(\phi)$. If $\phi = E(x,y)$, then $(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta) \models \phi$ iff $(\beta(x),\beta(y)) \in E^G$ iff, by (iii), $(g(z_{f(x)}), g(z_{f(y)})) \in E^G$ iff $(G,g,w) \Vdash @_{z_{f(x_1)}} \Diamond z_{f(x_2)} = tr_n^f(\phi)$. If $\phi = R_i(x_1,\ldots,x_h)$, then $(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta) \models \phi$ iff $(\beta(x_1),\ldots,\beta(x_n)) \in \mathbf{R}_i$ iff, by (iii), $(g(z_{f(x_1)}),\ldots,g(z_{f(x_n)})) \in \mathbf{R}_i$ iff, by (ii), $g(y^R_{f(x_1),\ldots,f(x_k)}) = g(t)$ iff $(G,g,w) \Vdash @_t y^R_{f(x_1),\ldots,f(x_k)} = tr_n^f(\phi)$.

— $\phi = \gamma \wedge \theta$ or $\phi = \neg\gamma$: These cases follow directly from the definition of tr_n^f and the inductive hypothesis.

— $\phi = \exists x\gamma$: In this case, $(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta) \models \phi$ iff there is a $v \in V$ such that $(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta\frac{x}{v}) \models \gamma$. By (i), there is a j be such that $v = z_j$. Hence we have $\beta\frac{x}{v}(y) = g(z_{f\frac{x}{j}(y)})$ for each first-order variable y. By inductive hypothesis we have, $(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta\frac{x}{v}) \models \gamma$ iff $(G,g,w) \Vdash tr_n^{f\frac{x}{j}}(\gamma)$ iff $(G,g,w) \Vdash \bigvee_{i=1}^n tr_n^{f\frac{x}{i}}(\gamma) = tr_n^f(\phi)$.

— $\phi = \forall x\gamma$: In this case, $(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta) \models \phi$ iff, for each $v \in V$, $(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta\frac{x}{v}) \models \gamma$. By (i), for each $v \in V$ there is a j such that $v = z_j$. Hence we have $\beta\frac{x}{v}(y) = g(z_{f\frac{x}{j}(y)})$ for each first-order variable y. By inductive hypothesis we have, for each $v \in V$, $(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta\frac{x}{v}) \models \gamma$ iff, for each $j \in \{1,\ldots,n\}$, $(G,g,w) \Vdash tr_n^{f\frac{x}{j}}(\gamma)$ iff $(G,g,w) \Vdash \bigwedge_{i=1}^n tr_n^{f\frac{x}{i}}(\gamma) = tr_n^f(\phi)$.

□

Proof of Lemma 8. Let $v \in V$ such that $v \neq g(t)$. For each $\mathbf{X}_1$ on V, let $v^{\mathbf{X}_1}_{i_1,\ldots,i_h} = g(t)$ if $(g(z_{i_1}),\ldots,g(z_{i_h}))$ and v otherwise. Then, for each $\mathbf{X}_1$ on V

there is an assignment $g^{\mathbf{X}_1}$ defined as

$$g^{\mathbf{X}_1} = g\frac{y^{X_1}_{1,\ldots,1}\cdots y^{X_1}_{i_1,\ldots,i_h}\cdots y^{X_1}_{n,\ldots,n}}{v^{\mathbf{X}_1}_{1,\ldots,1}\cdots v^{\mathbf{X}_1}_{i_1,\ldots,i_h}\cdots v^{\mathbf{X}_1}_{n,\ldots,n}}.$$

On the contrary, given an assignment g' we can find $\mathbf{X}_1$ such that $g' = g^{\mathbf{X}_1}$.

The assignment $g^{\mathbf{X}_1}$ can be described as:

$$g^{\mathbf{X}_1}(s) = \begin{cases} g(t) & \text{, if } s = y^{X_1}_{i_1,\ldots,i_h} \text{ and } (i_1,\ldots,i_h) \in \mathbf{X}_1; \\ v & \text{, for some } v \neq g(t)\text{, if } s = y^{X_1}_{i_1,\ldots,i_h} \text{ and } (i_1,\ldots,i_h) \notin \mathbf{X}_1; \\ g(s) & \text{, otherwise;} \end{cases}$$

It follows that $g^{\mathbf{X}_1}$ and $\mathbf{X}_1$ satisfies (i) and (iii) and

(ii') $g^{\mathbf{X}_1}(y^R_{i_1,\ldots,i_k}) = g^{\mathbf{X}_1}(t)$ iff $(g^{\mathbf{X}_1}(z_{i_1}),\ldots,g^{\mathbf{X}_1}(z_{i_k})) \in \mathbf{R}$ for each $R \in \{R_1,\ldots,R_m,X_1\}$.

Now, we proceed by induction on the size l of the prefix $Q_1X_1\ldots Q_lX_l$. If $l = 0$, then ϕ is first-order and the result follows immediately from Lemma 7. Suppose that $l > 0$. If $\phi = \exists X_1 \ldots Q_lX_l\psi$, then $(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta) \models \phi$ iff there is $\mathbf{X_1} \subseteq V^{r_1}$ such that

$$(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\mathbf{X}_1,\beta) \models Q_2X_2\ldots Q_lX_l\psi$$

iff, by the inductive hypothesis, there is $g^{\mathbf{X}_1}$ such that

$$(G, g^{\mathbf{X}_1}, w) \Vdash T^n(Q_2X_2\ldots Q_lX_l\psi)$$

iff

$$(G, g\frac{y^{X_1}_{1,\ldots,1}\cdots y^{X_1}_{i_1,\ldots,i_h}\cdots y^{X_1}_{n,\ldots,n}}{v^{\mathbf{X}_1}_{1,\ldots,1}\cdots v^{\mathbf{X}_1}_{i_1,\ldots,i_h}\cdots v^{\mathbf{X}_1}_{n,\ldots,n}}, w) \Vdash T^n(Q_2X_2\ldots Q_lX_l\psi)$$

iff

$$(G,g,w) \Vdash E\downarrow y^{X_1}_{\overline{1}}.\ldots.E\downarrow y^{X_1}_{\overline{n}}.T^n(Q_2X_2\ldots Q_lX_l\psi) = T^n(\phi).$$

If $\phi = \forall X_1\ldots Q_lX_l\psi$, then

$$(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\beta) \models \phi$$

iff, for all $\mathbf{X_1} \subseteq V^{r_1}$,

$$(G,\mathbf{R}_1,\ldots,\mathbf{R}_m,\mathbf{X}_1,\beta) \models Q_2X_2\ldots Q_lX_l\psi$$

iff, by the inductive hypothesis, for all $g^{\mathbf{X}_1}$,

$$(G, g^{\mathbf{X}_1}, w) \Vdash T^n(Q_2X_2\ldots Q_lX_l\psi)$$

iff, for all $v^{\mathbf{X}_1}_{1,\ldots,1}\ldots v^{\mathbf{X}_1}_{i_1,\ldots,i_h}\ldots v^{\mathbf{X}_1}_{n,\ldots,n}$,

$$(G, g\frac{y^{X_1}_{1,\ldots,1}\cdots y^{X_1}_{i_1,\ldots,i_h}\cdots y^{X_1}_{n,\ldots,n}}{v^{\mathbf{X}_1}_{1,\ldots,1}\cdots v^{\mathbf{X}_1}_{i_1,\ldots,i_h}\cdots v^{\mathbf{X}_1}_{n,\ldots,n}}, w) \Vdash T^n(Q_2X_2\ldots Q_lX_l\psi)$$

iff $(G,g,w) \Vdash A\downarrow y^{X_1}_{\overline{1}}.\ldots.A\downarrow y^{X_1}_{\overline{n}}.T^n(Q_2X_2\ldots Q_lX_l\psi) = T^n(\phi)$. □

Proof of Theorem 1.

$$(G, g, w) \Vdash \downarrow t.E \downarrow z_1.\dots E \downarrow z_n.\left[\left(\bigwedge_{1\le i<j\le n} @_{z_i}\neg z_j\right) \wedge T^n(\psi)\right]$$

iff

$$(G, g\frac{t}{w}, w) \Vdash E \downarrow z_1.\dots E \downarrow z_n.\left[\left(\bigwedge_{1\le i<j\le n} @_{z_i}\neg z_j\right) \wedge T^n(\phi)\right]$$

iff there are $v_1, \dots, v_n \in V$ such that

$$(G, g\frac{tz_1 \dots z_n}{wv_1 \dots v_n}, w) \Vdash \left[\left(\bigwedge_{1\le i<j\le n} @_{z_i}\neg z_j\right) \wedge T^n(\phi)\right]$$

iff there are $v_1 \neq \dots \neq v_n \in V$ such that

$$(G, g\frac{tz_1 \dots z_n}{wv_1 \dots v_n}, w) \Vdash T^n(\phi)$$

iff, by Lemma 8, $G \Vdash \phi$. □

Proof of Theorem 3. Let ψ be a second-order formula expressing $\mathcal{G}$. Let

$$\theta_n = A \downarrow z_1.\dots A \downarrow z_n.\left[\bigwedge_{1\le i<j\le n} @_{z_i}\neg z_j \to A \downarrow z.\left(\bigvee_{1\le i\le n} @_{z_i} z\right)\right].$$

The sentence θ_n says that there are at most n vertices in the frame. We define ϕ_n as:

$$\phi_n = \theta_n \to \downarrow t.E \downarrow z_1.\dots E \downarrow z_n.\left[\left(\bigwedge_{1\le i<j\le n} @_{z_i}\neg z_j\right) \wedge T^n(\psi)\right].$$

Let $G \in \mathcal{G}$. Let g be any assignment of state variables and w be any point in G. If $G \not\Vdash \theta_n$, then $G \Vdash \phi_n$. It follows that $G \Vdash \phi_n$ for each $n \neq |G|$. Hence, $G \Vdash \Phi$ iff $G \Vdash \phi_{|G|}$. Let $|G| = n$. Then $(G, g, w) \Vdash \phi_n$ iff

$$(G, g, w) \Vdash \downarrow t.E \downarrow z_1.\dots E \downarrow z_n.\left[\left(\bigwedge_{1\le i<j\le n} @_{z_i}\neg z_j\right) \wedge T^n(\psi)\right]$$

iff, by Theorem 1, $G \in \mathcal{G}$. □

Proof of Theorem 5. Let ψ be a second-order formula expressing $\mathcal{G}$. Let $\mathbf{Q}$ be the prefix $(\Box\Box^{-1})^n \downarrow z_1.\dots(\Box\Box^{-1})^n \downarrow z_n.$ and let

$$\theta_n = \mathbf{Q}\left[\bigwedge_{1\le i<j\le n} @_{z_i}\neg z_j \to (\Box\Box^{-1})^n \downarrow z.\left(\bigvee_{1\le i\le n} @_{z_i} z\right)\right], \text{ and}$$

$$\psi_n = \downarrow t.(\Diamond\Diamond^{-1})^n \downarrow z_1.\dots(\Diamond\Diamond^{-1})^n \downarrow z_n.\left[\left(\bigwedge_{1\le i<j\le n} @_{z_i}\neg z_j\right) \wedge T^n(\psi)\right].$$

Let $\phi_n = \theta_n \to \psi_n$. The remaining of the proof is similar to the proof of Theorem 2. □

Proof of Theorem 6. Let $\mathcal{G}$ be an NP-complete graph property. Let ϕ be an $\exists$SO sentence which express $\mathcal{G}$. By Fagin's Theorem [Fag74], such sentence exists. Let G be a graph. Let $T^{|G|}(\phi)$ as defined in Definition 9. It is easy to see that $T^{|G|}(\phi)$ can be constructed from ϕ in time polynomial in $|G|$. Now, $(G, T^{|G|}(\phi))$ is an instance of the model checking for FHL$\setminus \downarrow \Box \downarrow$. By Theorem 3, the model checker returns *true* for $(G, T^{|G|}(\phi))$ iff $G \in \mathcal{G}$. Hence, the model checking for FHL$\setminus \downarrow \Box \downarrow$ is hard for NP. □

Proof of Theorem 7. Analogous to the proof of Theorem 6 above, since for each graph property in Σ_i^p (resp. Π_i^p) can be expressed by a SO sentence in Σ_i^1 (resp. Π_i^1), and $T^n(\phi)$ can always be constructed in time polynomial in n, for a fixed $\phi \in SO$. □

Proof of Theorem 8. We proceed by induction on i. In [tCF05], it is shown that the model checking for FHL$\setminus \downarrow \Box \downarrow$, which contains σ^1, is in NP. It follows that π^1 is in co-NP. Now, let $\overline{q}\phi$ be a sentence in σ^{i+1}, where ϕ is in π^i. For the sake of simplicity, we disconsider the modalities $\Diamond^{-1}$ and E, but the proof is analogous. It follows that $\overline{q}$ has the form

$$\overline{q} = \Diamond^{k_1} \downarrow x_1.\Diamond^{k_2} \downarrow x_2.\dots\Diamond^{k_m} \downarrow x_m.\Diamond^{k_{m+1}}.$$

Let M be a finite model, g be an assignment of state variables and w a point in W. By inductive hypothesis, suppose that the model checking problem for π^i is in Π_i^p. We can use non-deterministic Turing machine to existentially guess values v_j for x_j among the points in W which are reachable in $\sum_{i=1}^{j} k_i$ steps from w, with respect to the accessibility relation R, in polynomial non-deterministic time, and we can existentially guess points w' reachable in $\sum_{i=1}^{m+1} k_i$ steps from w in polynomial non-deterministic time also. Finally, we can use an oracle for the model checking of π^i with the input

$$(M, g\frac{v_1\dots v_m}{x_1\dots x_m}, w'), \phi).$$

By inductive hypothesis, such an oracle is in Π_i^p. As the existential guesses initially performed can be made in (existential) non-deterministic polynomial time, the model checking for σ^{i+1} is in Σ_{i+1}^p.

The proof is analogous for the model checking of π^{i+1}. □