AN ALGEBRAIC ALGORITHM FOR THE RESOLUTION OF SINGULARITIES OF FOLIATIONS

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ABSTRACT. We propose an algorithm that uses Gröbner bases to compute the resolution of the singularities of a foliation of the complex projective plane.

1. Introduction

Given the importance of the blowup morphism in algebraic geometry, it is not surprising to find that it also plays a key rôle in the theory of holomorphic foliations. The seminal result in this direction was proved by A. Seidenberg in [8]. He showed that every foliation defined over a surface can be resolved by blowup into another one whose singularities are reduced. See section 2 for the definitions of all the technical terms on blowup of foliations used in the paper.

Although the existence of a resolution is what one most often needs, it is sometimes necessary to compute the blowups step by step. This turns out to be a very laborious process, unless the foliation is very simple, and it immediately suggests the idea of programming a computer to perform the blowups and compute the resolution. At first sight it seems that this should be quite straightforward. However, a careful consideration of the problem shows that one must isolate the singularities before applying the blowup. Since the coordinates of the singularities will not be rational numbers, except in very special cases, the program seems to call for a combination of algebraic and numerical methods. This, of course, suffers from the usual problems caused by approximating a singular point.

In this paper we propose a completely algebraic approach to the problem, based on the fact that singularities with similar properties will behave similarly under blowup. This enables us to handle the singularities in batches, so that all computations turn out to be exact.

In order to make it more precise, let $\mathcal{F}$ be a saturated foliation of the complex projective plane defined by a 1-form with rational coefficients. Since $\mathcal{F}$ has a finite number of singularities we may assume, without loss of generality, that they all belong to the open set $z \neq 0$. Identifying this open set with $\mathbb{C}^2$ in the usual way, we have that the x-coordinates of the singularities of $\mathcal{F}$ will be roots of a polynomial f with rational coefficients.

Let d be a vector field that defines $\mathcal{F}$ in $z \neq 0$. The key to the algorithm is the observation that, in order to compute the blowup of d at a singular point p it is enough to know (1) the algebraic multiplity m of d at p, and (2) whether the m-jet

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of d at p is a multiple of the Euler vector field. Therefore, the x-coordinates of two singularities of d for which (1) and (2) do *not* coincide must be roots of different irreducible factors of f.

We use the same strategy to sort out the reduced singularities from the nonreduced ones at every stage of the resolution. This leads to two algorithms, called `StricTransform` and `reduced`, that we have combined to get the full resolution. The `resolutionf` algorithm returns information relative to the algebraic multiplicity of the singularities, the invariance of the exceptional divisor, together with the complete resolution tree.

These procedures have been implemented in the library `resfolia.lib` of the computer algebra system SINGULAR [4]. This library also contains a procedure called `hasDicritical` that checks whether the given foliation is, or is not, dicritical. This allows one to make effective use of the bound on the degree of an algebraic solution of a foliation given in [1]. More details on the implementation are given in section 5.

2. FOLIATIONS

In this section we discuss the basic facts about foliations of the complex projective plane in a way that is suitable for the applications of the forthcoming sections.

Let x, y and z be homogeneous coordinates in $\mathbb{P}^2$, the complex projective plane. A *holomorphic foliation* of $\mathbb{P}^2$ is defined by a 1-form $\omega = Adx + Bdy + Cdz$, where A, B and C are nonconstant homogeneous polynomials of the same degree that satisfy the identity $xA + yB + zC = 0$.

Let U_z be the open set of $\mathbb{P}^2$ defined by $z \neq 0$ and let α be the dehomogeneization of ω with respect to z. Restricting the foliation of $\mathbb{P}^2$ defined by ω to U_z, we obtain the foliation of $\mathbb{C}^2$ defined by α. Conversely, if $\pi_z : U_z \to \mathbb{C}^2$ is the map given by $\pi_z[x : y : z] = (x/z, y/z)$, then $\omega = z^k \pi_z^*(\alpha)$, where k is chosen so as to clear the poles of $\pi_z^*(\alpha)$.

From now on we deal only with a foliation of $\mathbb{C}^2$ defined by a 1-from $\alpha = adx + bdy$, where $a, b \in \mathbb{C}[x, y]$. Moreover, we assume that α is *saturated*, which means $\gcd(a, b) = 1$. A *singularity* of α is a common zero of a and b. The set of all the singularities of α is denoted by $\mathrm{Sing}(\alpha)$. It follows from Bézout's theorem that this is a finite set, because we are assuming that α is saturated.

Since the blowup is a local construction, we may restrict ourselves to a neighbourhood V of one of the singularities p of α. Moreover, we assume that p is the only singularity of α contained in V. In order to simplify the notation we will choose the coordinates so that $p = (0, 0)$. The *blowup* of V with centre at p is the surface

$$\mathrm{B}_p(V) = \{((x, y), [u : v]) \in V \times \mathbb{P}^1 : xv = yu\}.$$

The *blowup map* is the morphism $\phi : \mathrm{B}_p(V) \to V$ given by $\phi((x, y), [u : v]) = (x, y)$. Note that $\mathrm{B}_p(V)$ is the union of two open affine sets isomorphic to V; namely, the sets given by $u \neq 0$ and $v \neq 0$, respectively. For instance, if $v \neq 0$, then the isomorphism maps $(u, y) \in V$ to $((yu, y), [u : 1]) \in \mathrm{B}_p(V)$. Identifying $\mathrm{B}_p(V)_{|v\neq 0}$ with V in this way, the map ϕ restricted to $\mathrm{B}_p(V)_{|v\neq 0}$ becomes $\phi_v(u, y) = (yu, y)$. Similarly, we may identify $\mathrm{B}_p(V)_{|u\neq 0}$ in a natural way with V such that $\phi_u(x, v) = (x, xv)$ is the restriction of ϕ to $\mathrm{B}_p(V)_{|u\neq 0}$. Note also that, outside the closed set

$$E = \phi^{-1}(p) = \{(p, [u : v]) : u, v \in \mathbb{C}^2\},$$

the map ϕ is an isomorphism. More precisely, ϕ maps $\mathrm{B}_p(V) \setminus E$ isomorphically onto $V \setminus \{p\}$.

The blowup of the foliation of V determined by $\alpha|_V$ is defined in terms of the inverse image of α under ϕ. In order to give an explicit formula we consider the foliation restricted to the open set $v \neq 0$. The case $u \neq 0$ may be dealt with similarly.

First, we need some notation. Let $f \in \mathbb{C}[x, y]$, and denote by $f = f_0 + f_1 + \cdots + f_m$ the decomposition of f into homogeneous components. Thus f_j is a homogeneous polynomial of degree j in x and y. Write

$$\alpha_k = a_k dx + b_k dy.$$

The *algebraic multiplity* of $\alpha = adx + bdy$ at $p = (0, 0)$ is the smallest $k \geq 0$ such that $\alpha_k \neq 0$. In particular, the algebraic multiplicity of a form η at $p \in \mathbb{C}^2$ is zero if and only if η is nonsingular at p.

A simple computation shows that

$$\phi_v^*(\alpha) = ya(uy, y)du + (b(uy, y) + ua(uy, y))dy. \tag{2.1}$$

Setting $\Delta(x, y) = xa + yb$, and taking into account that $a(0) = b(0) = 0$, it follows that $\phi_v^*(\alpha) = y\beta$, where

$$\beta = a(uy, y)du + \frac{\Delta(uy, y)}{y^2} dy,$$

is a 1-form with polynomial coefficients. Suppose now that $k > 0$ is the algebraic multiplicity of α at p. Then

$$\Delta(uy, y) = y^{k+1}(\Delta_{k+1}(u, 1) + \sum_{j=k+2}^{s} y^{j-(k+1)} \Delta_j(u, 1)).$$

Thus we can rewrite (2.1) as

$$\phi_v^*(\alpha) = y^k (ya_k(u, 1)du + \Delta_{k+1}(u, 1)dy) + y^{k+1}\widehat{\alpha},$$

where $\widehat{\alpha}$ is a 1-form in u and y whose algebraic multiplicity is at least $k + 1$. The *strict transform* of α under the pullback ϕ_v is the saturation of $\phi_v^*(\alpha)$, which is equal to

$$\frac{\beta}{y^{k-1}} = ya_k(u, 1)du + \Delta_{k+1}(u, 1)dy + y\widehat{\alpha} \quad \text{if} \quad \Delta_{k+1} \neq 0$$

or to

$$\frac{\beta}{y^k} = a_k(u, 1)du + \Delta_{k+2}(u, 1)dy + y\widetilde{\alpha} \quad \text{if} \quad \Delta_{k+1} = 0;$$

where $\tilde{\alpha}$ is a 1-form of multiplicity greater than or equal to $k + 2$. Note that $\Delta_{k+1} = 0$ if and only the exceptional divisor $y = 0$ is not invariant under the strict transform of α. In this case we say that α is *dicritical* at 0.

Since we will have to iterate this process, it is necessary to determine the singularities of the blowup of $\alpha|_V$ under ϕ_v. However, ϕ_v is an isomorphism outside p, which is the only singularity of α in V. Therefore, the singularities of the strict transform of $\alpha|_V$ under ϕ_v must belong to the exceptional divisor $E = \phi^{-1}(p)$. In particular, all the singularities of this strict transform have their y-coordinate equal to 0. Hence, the singularities are given by

$$\begin{aligned} \Delta_{k+1}(u, 1) = 0 \ &\text{ when the singularity is nondicritical} \\ a_k(u, 1) = \Delta_{k+2}(u, 1) = 0 \ &\text{ when the singularity is dicritical} \end{aligned} \tag{2.2}$$

This is summed up in Table 1, for future reference. As before, we are assuming that α has algebraic multiplicity k at the origin.

Case	Δ_{k+1}	Strict transform	Singularities
dicritical	0	β/y^k	$y = a_k(u,1) = \Delta_{k+2}(u,1) = 0$
nondicritical	$\neq 0$	β/y^{k-1}	$y = \Delta_{k+1}(u,1) = 0$

TABLE 1. Blowup with respect to $v \neq 0$

Up to now we have only computed the blowup of α with respect to the open set $v \neq 0$ of $\mathrm{B}_p(V)$. We must now repeat the same computation with respect to $u \neq 0$. On doing that we will find that the *strict transform* of α under the pullback ϕ_u is equal to $x\eta$, where

$$\eta = \frac{\Delta(x, xv)}{x^2} dx + b(x, xv)dv.$$

Assuming now that α has algebraic multiplicity k at the origin, we have the following table:

Case	Δ_{k+1}	Strict transform
dicritical	0	η/x^k
nondicritical	$\neq 0$	η/x^{k-1}

TABLE 2. Blowup with respect to $u \neq 0$

This time we have not added the singularities to the table because the open set $u \neq 0$ can only contribute one new singularity to the strict transform of the foliation determined by α in V; namely $x = v = 0$. Indeed, this is the only singularity that belongs to the exceptional divisor and does not have nonzero v-coordinate.

Although we cannot completely remove the singularities of a foliation $\mathcal{F}$ by blowup, we can simplify them so that the behaviour of the local holomorphic solutions of $\mathcal{F}$ can be more easily described.

Suppose that p is a singularity of a 1-form $\alpha = adx + bdy$ of $\mathbb{C}^2$. Let λ_1 and λ_2 be the eigenvalues of the 1-jet $j_1(\alpha)(p)$ of the vector field $b\partial/\partial x - a\partial/\partial y$ at p. Then p is a *reduced* singularity of α if $\lambda_2 \neq 0$ and either

(1) $\lambda_1 = 0$, or
(2) λ_1/λ_2 is not a positive rational number.

Given a singular holomorphic foliation of $\mathbb{C}^2$, we can always [8] resolve its singularities into reduced singularities by a succession of blowups. More precisely, if $p \in \mathbb{C}^2$ is a singular point of α there is a resolution

$$S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = \mathbb{C}^2,$$

such that

(1) π_i is a blowup of S_{i-1} with centre p_{i-1};
(2) $\alpha_0 = \alpha$;
(3) $p_0 = p$;

(4) $\mathcal{F}_0 = \mathcal{F}$;
(5) if α_i is the strict transform of α_{i-1} under π_i, then p_i is a singular point of α_i for $i = 1, \ldots, n-1$; and,
(6) if $q \in S_n$ projects to p under $\pi_1\pi_2 \cdots \pi_n$ then, either q is a reduced singularity of α_n, or α_n is nonsingular at q.

In order to determine when to stop the resolution process we must be able to detect whether a given singularity of a 1-form is reduced. Since the problem is local, we may assume that the singularity is $p = (0,0)$ and that the polynomial 1-form $\alpha = adx + bdy$ has a singularity at p. Let λ_1 and λ_2 be the eigenvalues of

$$j_1(\alpha) = \begin{bmatrix} \partial b/\partial x & \partial b/\partial y \\ -\partial a/\partial x & -\partial a/\partial y \end{bmatrix}, \tag{2.3}$$

at p. Denote by $t(p)$ and $d(p)$ the trace and determinant of this matrix at p.

If $\lambda_1 = \lambda_2 = 0$, the singularity is nonreduced. This is easy to detect because the eigenvalues are both zero if and only if $t(p) = d(p) = 0$. Similarly, $\lambda_1 = 0$ and $\lambda_2 \neq 0$ can only occur if $d(p) = 0$ but $t(p) \neq 0$. However, in this case p is a reduced singularity. We are left only with the case $\lambda_1\lambda_2 \neq 0$. But,

$$\frac{t(p)^2}{d(p)} = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1\lambda_2} = \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + 2.$$

Writing $q = \lambda_1/\lambda_2$ we find that

$$d(p)q^2 + (2d(p) - t(p)^2)q + d(p) = 0. \tag{2.4}$$

Let

$$h(x, y, z) = d(x, y)w^2 + (2d(x, y) - t(x, y)^2)w + d(x, y).$$

In this case p is reduced if and only if $h(p, w) = 0$ has positive rational roots. Note that we can also detect that one of the eigenvalues of α at p is zero by looking at h. Indeed, in this case $h(p, w)$ is a multiple of w.

In the next two sections we discuss strategies, based on the use of Gröbner basis, to blowup a 1-form at its singularities, and to determine whether a given singularity is reduced. These can then be combined into a full-fledged resolution algorithm.

3. Blowing up a 1-form

Let $a = a(x, y)$ and $b = b(x, y)$ be polynomials in the variables x and y, with coefficients in $\mathbb{Q}[x_0, \ldots, x_n]$, and let I be a 0-dimensional ideal of $\mathbb{Q}[x_0, \ldots, x_n][x, y]$ which contains a and b. The points of $\mathcal{Z}(I) \subseteq \mathbb{C}^{n+2}$ will be called the *singularities modulo* I of the form $\alpha = adx + bdy$. Note that such a point can be written in the form $(p, \tilde{x}, \tilde{y})$, where $p \in \mathbb{C}^{n+1}$ is a zero of $I \cap \mathbb{Q}[x_0, \ldots, x_n]$. Thus, if α_p is the form obtained specializing $(x_0, \ldots, x_n)$ to the point p, then $(\tilde{x}, \tilde{y})$ a singularity of α_p.

In order to simplify the formulae, it is convenient to work with the pullback $\tilde{\alpha}$ of α under the translation that takes $(0,0)$ to (x_{n+1}, y_{n+1}). If

$$\tilde{a}(x, y) = a(x + x_{n+1}, y + y_{n+1}) \text{ and } \tilde{b}(x, y) = b(x + x_{n+1}, y + y_{n+1}),$$

then $\tilde{\alpha} = \tilde{a}(x, y)dx + \tilde{b}(x, y)dy$. Note that $\tilde{a}$ and $\tilde{b}$ are polynomials in the variables x and y, with coefficients in $\mathbb{Q}[x_0, \ldots, x_n, x_{n+1}, y_{n+1}]$.

We will assume, throughout this section, that I is a radical ideal of dimension zero, in general position with respect to x. Let $\tilde{I}$ the ideal obtained replacing x by x_{n+1} and y by y_{n+1} in I.

From now on we assume that the monomials of $\mathbb{Q}[x_0, \dots, x_{n+1}, y_{n+1}]$ are ordered by the lexicographical order with $x_0 > x_1 > \dots > y_{n+1} > x_{n+1}$. By the hypothesis on I, and [6, Theorem 3.7.25, p. 257], intersecting a reduced Gröbner basis of $\tilde{I}$ with $\mathbb{Q}[x_{n+1}, y_{n+1}]$ we get a set of the form

$$\{y_{n+1} - g(x_{n+1}), f(x_{n+1})\}.$$

Thus we may replace y_{n+1} with $g(x_{n+1})$ in $\tilde{\alpha}$, which means that the singularities of $\tilde{\alpha}$ modulo $\tilde{I}$ are completely determined by the ideal $L \subset \mathbb{Q}[x_0, \dots, x_n, x_{n+1}]$ obtained by replacing y_{n+1} by $g(x_{n+1})$ in $\tilde{I}$. Moreover, since I is radical and has dimension zero, so is $L = \tilde{I} \cap \mathbb{Q}[x_0, \dots, x_{n+1}]$. Therefore, we may reset the notation, as follows

> $\alpha = adx + bdy$ is a 1-form with $a, b \in \mathbb{Q}[x_0, \dots, x_n, x_{n+1}][x, y]$, and I is a 0-dimensional radical ideal of $\mathbb{Q}[x_0, \dots, x_n, x_{n+1}]$ that is in general position with respect to x, and contains $a(0,0)$ and $b(0,0)$. Moreover,
> $$(f) = I \cap \mathbb{Q}[x_{n+1}].$$

We will now compute explicit formulae for the blowup of α at $p = (0,0)$ with respect to the coordinate system defined, in the notation of section 2, by $v \neq 0$. In order to simplify the notation we denote ϕ_v simply by ϕ. Thus $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ is given by $\phi(u, y) = (uy, y)$. Hence, $\phi^*(\alpha)$ is given by equation (2.1). If $\Delta = xa + yb$, then

$$\frac{\Delta(uy, y)}{y} = b(uy, y) + ua(uy, y).$$

Moreover, since $a(0,0) \equiv b(0,0) \equiv 0 \pmod{I}$, it follows that y divides $b(uy, y) + ua(uy, y)$ modulo I. Therefore,

$$\beta_1 = a(uy, y)du + \frac{\Delta(uy, y)}{y^2}dy,$$

is well defined modulo I. In fact, since the coefficients of degree 1 of $\Delta(uy, y)$ are always zero modulo I, we may as well delete them. Indeed, from now on, we always adopt the policy of deleting the coefficients that we know to be always zero modulo the ideal that defines the singularities. To make this task easier, we introduce the following notation. Let $F \in \mathbb{Q}[x_0, \dots, x_{n+1}][x, y]$. Using the decomposition of F in its homogenous components with respect to the variables x and y, we have that

$$F(uy, y) = \sum_{j=0}^{s} y^j F_j(u, 1).$$

Then $\tau_k^y(F)$ will be the polynomial

$$\frac{\sum_{j=k}^{s} y^j F_j(u, 1)}{y^k} = \sum_{j=k}^{s} y^{j-k} F_j(u, 1).$$

τ_k^x is analogously defined.

Denote by $\texttt{coeff}(F)$ the set of coefficients of $F \in \mathbb{Q}[x_0, \dots, x_n, x_{n+1}][x, y]$ with respect to x and y. Thus $\texttt{coeff}(F) \subset \mathbb{Q}[x_0, \dots, x_n, x_{n+1}]$. Let

$$\texttt{coeff}(\alpha) = \{\texttt{coeff}(a), \texttt{coeff}(b)\}.$$

If

$$(\texttt{coeff}(\alpha_1), I) = (\texttt{coeff}(\Delta_2), I) = (1),$$

then all the singularities of α are nondicritical and have algebraic multiplicity one. However, if the algebraic multiplicity of α at any of its singularities is greater than one, or the singularity is dicritical, then we still have to factor a power of y from β. In order to do this we must be able to sort the singularities of α by their multiplicities, taking also into account whether they are dicritical or nondicritical. We do this using a strategy that relies on the following result.

Proposition 3.1. *Let I and J be 0-dimensional ideals of $\mathbb{Q}[x_0, \dots, x_n, x_{n+1}]$. Assume that J is a radical ideal in general position with respect to x_{n+1}, and let q be the generator of $J \cap \mathbb{Q}[x_{n+1}]$. If q is irreducible and $\mathcal{Z}(I) \cap \mathcal{Z}(J) \neq \emptyset$, then $I \equiv 0 \pmod{J}$.*

Proof. Since $\mathcal{Z}(I) \cap \mathcal{Z}(J) \neq \emptyset$ it follows that $(I, J) \subsetneq (1)$. Therefore,

$$(q) \subseteq (I, J) \cap \mathbb{Q}[x_{n+1}] \subsetneq (1).$$

However, q is irreducible in $\mathbb{Q}[x_{n+1}]$, so that $(q) = (I, J) \cap \mathbb{Q}[x_{n+1}]$. Since J is in general position with respect to x_{n+1}, it follows that $\mathcal{Z}(I, J) = \mathcal{Z}(J)$. Hence, by the Nullstellensatz,

$$J \subseteq \sqrt{(I, J)} = \sqrt{J} = J,$$

because J is radical. It follows that $J = \sqrt{(I, J)}$. In particular, $I \subseteq J$, which implies the required result. □

Now let q be an irreducible factor of f in $\mathbb{Q}[x_{n+1}]$, and consider the ideal $J = (I, q)$. Recall that I is radical, 0-dimensional, and in general position with respect to x_{n+1}, and that q is irreducible. Thus, J is radical and 0-dimensional. According to Proposition 3.1, if there is a singularity of α modulo J whose algebraic multiplicity is greater than k, then $(\texttt{coeff}(\alpha_k)) \equiv 0 \pmod{J}$. In particular, all these singularities have the same multiplicity. Therefore, the algebraic multiplicity of the singularities of α modulo J is equal to the smallest m for which $(\texttt{coeff}(\alpha_m)) \not\equiv 0 \pmod{J}$, which is easy to find by a simple search.

Before we proceed to compute the strict transform of α at its singularities modulo J, we must find if any of them are dicritical. Once again, it follows from Proposition 3.1, that the singularities of α modulo J are either all of them dicritical, or nondicritical. Moreover, if these singularities have multiplicity m then,

$$\texttt{coeff}(\Delta_{m+1}) \equiv 0 \pmod{J} \text{ when the singularities are dicritical}$$
$$\texttt{coeff}(\Delta_{m+1}) \not\equiv 0 \pmod{J} \text{ when the singularities are nondicritical}$$

Assuming that the singularities of α modulo J have multiplicity m, we sum up all this in Table 3.

Singularity type	$\texttt{coeff}(\Delta_{m+1}) \pmod{J}$	Strict Transform
dicritical	0	$\tau_m^y(\beta)$
nondicritical	$\neq 0$	$\tau_{m-1}^y(\beta)$

TABLE 3. Strict transform when $v \neq 0$

We now turn to the problem of detecting the singularities at which α is nonreduced. We keep the notation that we have been using in the previous discussion.

Denote by t the trace and d the determinant of the 1-jet $j_1(\alpha)$ at $(0,0)$, defined in (2.3). Note that $t, d \in \mathbb{Q}[x_0, \dots, x_n, x_{n+1}]$.

The first step consists in isolating the degenerate singularities of α, which correspond to the zeroes of (I, t, d). Since this ideal is in general position with respect to x_{n+1}, the degenerate singularities of α are those whose x_{n+1}-coordinate is a root of $\gamma_1 = 0$, where γ_1 is the generator of $\sqrt{(I,t,d)} \cap \mathbb{Q}[x_{n+1}]$. If α does not have any degenerate singularities then $\gamma_1 = 1$.

Turning now to the nondegenerate singularities, let

$$h = dw^2 + (2d - t^2)w + d \in \mathbb{Q}[x_0, \dots, x_n, x_{n+1}][w],$$

and compute the generator $m(w)$ of $\sqrt{(I,h)} \cap \mathbb{Q}[w]$. Then every characteristic exponent of α at one of its singularities is a root of $m(w)$, a one variable polynomial. Note that if an eigenvalue of α vanishes at one of its singularities, then 0 will be a root of $m(w)$.

The analysis at the end of section 2 shows that α has a nondegenerate and nonreduced singularity modulo I if and only if

$$-\frac{c_2}{c_1} = \frac{-\ell(0)}{\ell(1) - \ell(0)} > 0,$$

where $\ell = c_1 w + c_2$ is a linear factor of $m(w)$ over $\mathbb{Q}$. Moreover, if ℓ is such a factor then the x_{n+1}-coordinates of the singularities of α modulo I with characteristic exponent $-c_2/c_1$ are the zeroes of the ideal $(h(-c_2/c_1), I)$.

This allows us to sort the singularities of α that are reduced from those that are nonreduced. Let $r_1, \dots, r_s$ be the non-negative rational roots of $m(w)$. The nonreduced singularities of α are the zeroes of $(I, h(r_1) \cdots h(r_s)\gamma_1)$. To compute the reduced singularities we first find the generator γ_2 of $\sqrt{(I, h(r_1) \cdots h(r_s)\gamma_1)} \cap \mathbb{Q}[x_{n+1}]$. The reduced singularities are now the zeroes of $(I, f/\gamma_2)$, where f is the generator of $I \cap \mathbb{Q}[x_{n+1}]$.

4. The algorithms

In this section we describe in detail the three algorithms that will be combined to produce the resolution algorithm. The first of these algorithms prepares the output that will be used by the other two. The second sorts out the reduced singularities from those that are nonreduced, while the third one computes the actual blowup at the nonreduced singularities.

Algorithm: `precalc`

Input: a 1-form $\alpha = adx + bdy$ with $a, b \in \mathbb{Q}[x_0, \dots, x_n][x, y]$ and an ideal I of $\mathbb{Q}[x_0, \dots, x_n, x, y]$, which contains a and b.

Output: a list whose entries are:

- the 1-form α, with x translated by x_{n+1} and y translated by y_{n+1},
- the ideal obtained replacing x by x_{n+1} in $\sqrt{I} \cap Q[x_0, \dots, x_n, x]$, and
- the polynomial $\Delta = xa + yb$.

Step 1: If $\dim(I) > 0$ as an ideal in the ring $\mathbb{Q}[x_0, \dots, x_n][x, y]$ return `form is not saturated` and stop.

Step 2: Let $I = \sqrt{I}$.

STEP 3: Check whether I is in general position with respect to x. If it is not, let the user know that it is not, choose a map of the form

$$\phi(x_0,\dots,x_n,x,y) = (x_0,\dots,x_n,x+\sum_{i=0}^{n} c_i x_i + c_{n+1}y, y),$$

where $c_0,\dots,c_{n+1} \in \mathbb{Q}$, and replace I by $\phi^*(I)$ and α by $\phi^*(\alpha)$. Repeat this step until I is in general position. Note that $x_0,\dots,x_n$ play the role of constants, so far as α is concerned.

STEP 4: Make $x = x_{n+1}$ and $y = y_{n+1}$ in I.

STEP 5: Translate α by (x_{n+1}, y_{n+1}).

STEP 6: Compute $\Delta = xa + yb$.

STEP 7: Compute the reduced Gröbner basis G of I in $\mathbb{Q}[x_0,\dots,x_n][x_{n+1},y_{n+1}]$, with respect to `LEX`, assuming that $x_0 > \cdots > x_n > y_{n+1} > x_{n+1}$. Let

$$\{f, y_{n+1} - g(x_{n+1})\} = G \cap \mathbb{Q}[x_{n+1}, y_{n+1}].$$

STEP 8: Replace y_{n+1} by $g(x_{n+1})$ in I, α, and Δ. Thus α becomes a 1-form, and Δ a polynomial, over $\mathbb{Q}$ in the variables $x_0,\dots,x_n,x_{n+1},x,y$.

STEP 9: Return α, I, and Δ.

ALGORITHM: `reduced`

INPUT: a 1-form $\alpha = adx + bdy$ with $a,b \in \mathbb{Q}[x_0,\dots,x_{n+1}][x,y]$, $\Delta = xa+yb$, and an ideal I of $\mathbb{Q}[x_0,\dots,x_{n+1}]$, as in the output of `precalc`. Moreover, it is assumed that I is a radical ideal that is in general position with respect to x_{n+1}.

OUTPUT: a list $[I_{nr}, I_r, m, f]$, where I_{nr} and I_r are ideals of $\mathbb{Q}[x_0,\dots,x_n,x_{n+1}]$, m is a polynomial of $\mathbb{Q}[w]$, and f is a polynomial of $\mathbb{Q}[x_{n+1}]$, such that:

- $\mathcal{Z}(I_r)$ is the set of reduced singularities of α,
- m is the squarefree polynomial satisfied by the ratios of eigenvalues of the 1-jet of α at its nondegenerate singularities,
- $\mathcal{Z}(I_{nr})$ is the set of nonreduced singularities of α, and
- f is a generator of $\sqrt{I_{nr}} \cap \mathbb{Q}[x_{n+1}]$.

STEP 1: If $I = (1)$ then $I_{nr} = I_r = (1)$.

STEP 2: Let d and t be the determinant and trace at $(0,0)$ of the jacobian of $(b,-a)$ with respect to x and y. These are polynomials with coefficients in $\mathbb{Q}[x_0,\dots,x_n,x_{n+1}]$.

STEP 3: Let L be the ideal of $\mathbb{Q}[x_0,\dots,x_n,x_{n+1}]$ generated by I, t and d.

STEP 4: If $L \neq (1)$ compute the generator $\gamma_1 \neq 1$ of $\sqrt{L} \cap \mathbb{Q}[x_{n+1}]$. If $L = (1)$ put $\gamma_1 = 1$.

STEP 5: If $\gamma_1 = f$ return the list $[I, \{1\}, 1, f]$ and stop.

STEP 6: Let $h(w) = dw^2 + (2d - t^2)w + d$. This is a polynomial in w with coefficients in $\mathbb{Q}[x_0,\dots,x_n,x_{n+1}]$.

STEP 7: Consider the ideal $(I, h, f/\gamma_1)$ of $\mathbb{Q}[x_0,\dots,x_n,x_{n+1}][w]$, and compute the generator m of $\sqrt{(I,h,f/\gamma_1)} \cap \mathbb{Q}[w]$.

STEP 8: Factorize m in $\mathbb{Q}[w]$.

STEP 9: For each linear factor ℓ of m on $\mathbb{Q}[w]$ do:

If

$$q = \frac{-\ell(0)}{\ell(1) - \ell(0)} > 0$$

then $\gamma_1 = \gamma_1 \cdot h(q)$.

STEP 10: Let $I_{nr} = (I, \gamma_1)$.

STEP 11: Compute the generator γ_2 of $\sqrt{I_{nr}} \cap \mathbb{Q}[x_{n+1}]$, and let $I_r = (I, f/\gamma_2)$.

STEP 12: Return the list $[I_{nr}, I_r, m, \gamma_2]$.

ALGORITHM: `StrictTransform`

INPUT: a 1-form $\alpha = adx + bdy$ with $a, b \in \mathbb{Q}[x_0, \dots, x_{n+1}][x, y]$, $\Delta = xa + yb$, the ideal I_{nr} of $\mathbb{Q}[x_0, \dots, x_{n+1}]$, which defines the nonreduced singularities of α, and the polynomial f which generates $I_{nr} \cap \mathbb{Q}[x_{n+1}]$. Moreover, it is assumed that I is a radical ideal that is in general position with respect to x_{n+1}.

OUTPUT: a list whose elements are the strict transforms of the blowups of α at the singular points of α modulo I.

STEP 1: Compute

$$\beta = a(uy, y)du + \frac{\Delta(uy, y)}{y^2}dy \quad \text{and} \quad \eta = \frac{\Delta(x, xv)}{x^2}dx + b(x, xv)dv.$$

STEP 2: Initialize both $\mathcal{L}$ and $\mathcal{S}$ as empty lists. The list $\mathcal{L}$ will keep the blowups of α, while $\mathcal{S}$ retains information on the singularities of α.

STEP 3: If $f \in \mathbb{Q}$ stop, since there are no nonreduced singularities left to be blownup.

STEP 4: Factorize f as a polynomial in $\mathbb{Q}[x_{n+1}]$.

STEP 5: For every nonconstant irreducible factor q of f, do:

- Initialize $m = 1$ and consider the ideal (I, q) of $\mathbb{Q}[x_0, \dots, x_{n+1}]$.
- While $(\texttt{coeff}(\alpha_m)) \equiv 0 \pmod{(I, q)}$ let
  $$m = m + 1, \; \beta = \tau_1^y(\beta), \;\text{ and }\; \eta = \tau_1^x(\eta).$$
- Compute the reduction δ of $\texttt{coeff}(\Delta_{m+1})$ modulo (I, q).
- If $\delta = 0$, insert
  $$[\tau_1^y(\beta), (I, q, a_m(u, 1), \Delta_{m+2}(u, 1), y)] \;\text{ and }\; [\tau_1^x(\eta), (I, q, x, v)]$$
  into $\mathcal{L}$ and $[I, q, \texttt{dicritical}]$ into $\mathcal{S}$.
- If $\delta \neq 0$, insert
  $$[\beta, (I, q, \Delta_{m+1}(u, 1), y)] \;\text{ and }\; [\eta, (I, q, x, v)]$$
  into $\mathcal{L}$ and $[I, q, \texttt{nondicritical}]$ into $\mathcal{S}$.

STEP 6: Return $\mathcal{L}$ and $\mathcal{S}$.

We combined these three procedures in an algorithm, called `resolutionf`, whose input is a 1-form α with coefficients in $\mathbb{C}[x, y]$, and an ideal I of the same ring. This algorithm applies the sequence of three procedures described above recursively, in order to compute the resolution tree of α at $\text{Sing}(\alpha) \cap \mathcal{Z}(I)$. The output is a list which contains details about the various 1-forms encountered in the tree, their singularities, and the forms derived from them. Each leaf of the tree is labelled by a form β and an ideal J, such that either β has no singularities at $\mathcal{Z}(J)$, or

$\mathcal{Z}(J) \cap \mathrm{Sing}(\beta)$ consists only of reduced singularities. By [8] this algorithm always stops. The construction of the resolution tree applies a breadth-first strategy.

There is one final point that should be stressed. As implemented, these algorithms allow the user to restrict the blowups to any set of singularities that can be defined as the intersection of curves with rational coefficients. Indeed, one need only add the equations of these curves to the ideal I that appears as part of the input of all the algorithms.

5. Implementation and experimental tests

The algorithms discussed in the previous section have been implemented as part of a library (called `resfolia.lib`) designed to be used with the computer algebra system Singular (version 2-0-5), see [4] and [3]. Besides the four algorithms described in section 4, the library contains two other main algorithms, called `blowup` and `hasDicritical`. The former computes the blowup of a given form at each one of its singularities and stops. In other words, it computes only the height one nodes of the resolution tree. The latter algorithm computes the resolution tree looking for a dicritical singularity, and stops if one is found, or if the resolution comes to an end.

The algorithm `blowup` allows the user to choose both the variables required to represent the input and output forms. However, this would not be practicable in `resolutionf`, since it will typically produce trees with height well over 10. Thus, we have always used the same variables when defining a form and its blowups. More precisely, if β is a 1-form associated to a node of height k of the resolution tree then it is written in terms of x, y, and $x_0, \dots, x_{k-1}$. Of course, the x_i variables are only used to pin down the singularities of β; so these variables can be thought of as constants as far as the blowup is concerned. For the sake of consistence, we decided to keep these variables in the ring even when they define a rational number. This also helps to identify the height in the resolution tree of any given node. Note that the variables of the input form are always changed to x and y, independently of the variables originally chosen by the user. This convention may seen somewhat strange at first, but it has proved to be quite simple and memory saving.

Keeping to the notation of the previous paragraph, the strict transforms of β will be written in terms of the variables $x_0, \dots, x_k$ and the pair x and y, independently of the open set where the blowup has been computed. In order to allow the user to identify the corresponding open set we also return the formula that has been used to compute the blowup; see section 3.

The `hasDicritical` algorithm is essentially the same as `resolutionf`. The only difference is that, although it may be necessary to generate the whole resolution tree, we need not keep any of the nodes that correspond to forms that have already been blown up. Thus, we can save a lot of memory, which allows the algorithm to reach deeper into the resolution tree. Indeed, the most frequent cause of failure of `resolutionf` is lack of memory. This is not surprising since we have to store a growing number of forms and ideals defined over rings whose number of variables is also increasing.

There are very few complete resolution trees published in the literature on holomorphic foliations. Some of these appear in [7, Appendix B], and have been replicated using our algorithms. The one in Figure 1 corresponds to the 1-form

$$\omega_0 = (-80x^2 - 60xy + 80y)dx + (36x^2 - 32x - y)dy.$$

and was generated from the output of `resolutionf`. This output consists of a series of blocks of which the following is a typical example:

```
tag:  0-3-2
form:  (-160*x^3*y^2-381/4*x^2*y^2-1053/320*x*y^2
-51200/27*x*y-1701/81920*y^2 -80/9*y)*dx + (-80*x^4*y
-123/4*x^3*y-1971/1280*x^2*y- 25600/27*x^2-1701/81920*x*y
+80/9*x)*dy
open set:  (x,x*y)
nonreduced dicritical:
                 x,320*x1-9,27*x0-32,M(1)
nonreduced nondicritical:
                 1
reduced:
                 1
```

The `tag` describes the position of a given node in the tree, with respect to its ancestors. Thus `0-3-2` is a son of `0-3`, and grandson of `0`. The next line contains the 1-form that defines the foliation at this point of the resolution tree. Note that the form has been written in terms of the variables x and y, as explained above. After the form comes the formula used to compute the blowup; which allows us to identify the open set of $\mathbb{P}^1$ where the blowup was performed. This is followed by information on the singularities of the form. More precisely, we give the ideals that define the sets of nonreduced (dicritical and nondicritical), and reduced singularities. The ideals are written in terms of the variables `x0`, ... , `xk-1`, `x`, where k is the height of the vertex under consideration. In the example above, the vertex has height 3, and the ideal of nonreduced singularities (the only one that is nontrivial) is written in terms of `x0`, `x1` and `x`. Note that `27*x0-32` implies that $x_0 = 32/27$. However, as we have already mentioned, we chose to accumulate in the ideals all the polyomials when going from one height to the next, even when an `xi` variable corresponds only to a rational point. Finally, the `M(k)` after the ideal generators indicate that the sigularities defined by this ideal have multiplicity k in the given 1-form. Summing up, the origin is the only singularity of the 1-form

$$(-160x^3y^2 - \frac{381}{4}x^2y^2 - \frac{1053}{320}xy^2 - \frac{51200}{27}xy - \frac{1701}{81920}y^2 - \frac{80}{9}y)dx + \\ (-80x^4y - \frac{123}{4}x^3y - \frac{1971}{1280}x^2y - \frac{25600}{27}x^2 - \frac{1701}{81920}xy + \frac{80}{9}x)dy.$$

Moreover, it is a dicritical singularity of multiplicity one.

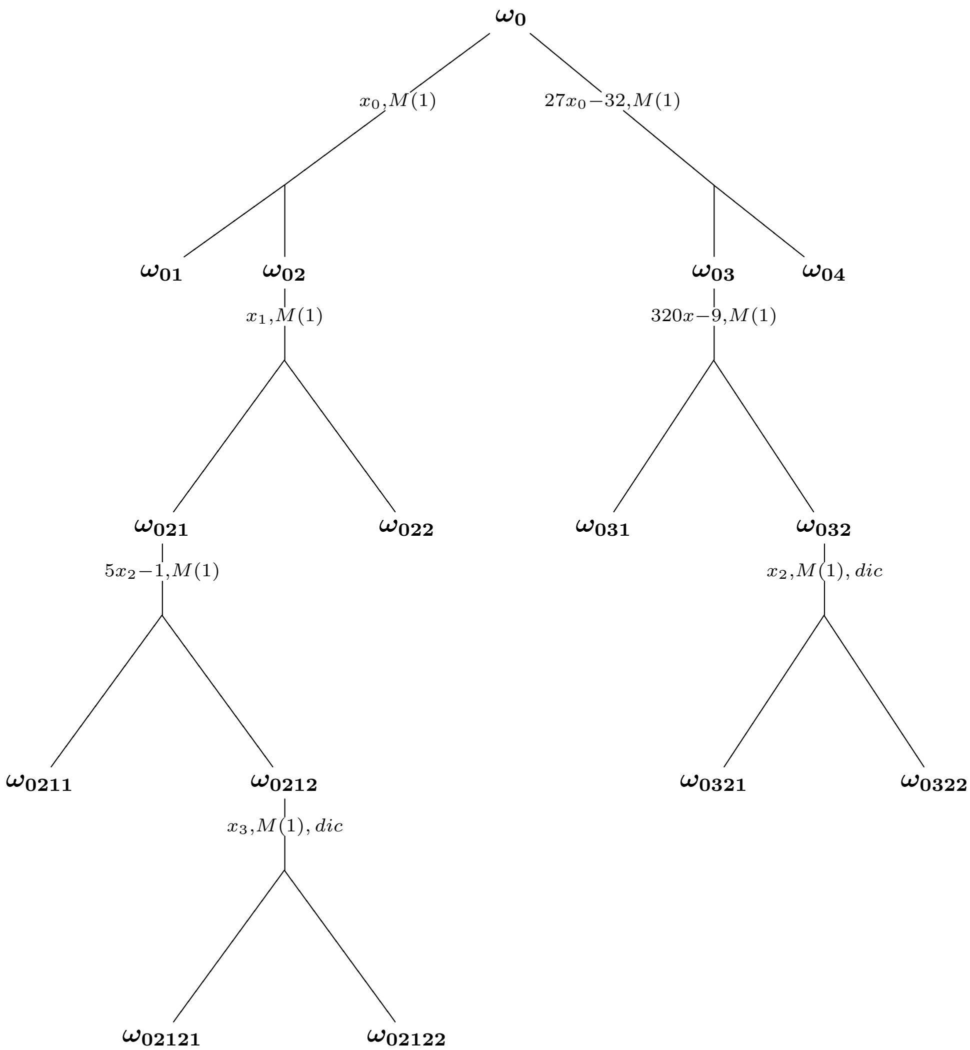

Figure 1

In order to simplify the representation of the resolution tree in Figure 1, we have used a number of conventions that we now explain. First, only one equation is given for the ideal that defines the singularity to be blownup at any given node v. Thus, the edge below a node v of height k is labelled only by a polyomial q_k in x_k. After all, the other equations already appear in the tree, since they are used to define the singularities of the forms that label the ancestors of v. More precisely, if the edges on a path from the root to v are labelled by the polynomials $q_0(x_0), \ldots, q_{k-1}(x_{k-1})$, then the singular set of the form β that is attached to v is given by

$$q_0(x_0) = \cdots = q_{k-1}(x_{k-1}) = q_k(x) = 0.$$

Note that x_k has been replaced by x in q_k to conform to our convention that β is always a form in x and y.

The only true nodes of the tree are those labelled by 1-forms. However, we also introduced "pseudonodes", where a given edge branches to indicate that the form is blownup at the two open sets of $\mathrm{B}_p(U_z)$. Every time there is such a branch point, the left hand node corresponds to the form blownup at the open set $v \neq 0$, and the right hand side to $u \neq 0$, in the notation of section 3.

The 1-forms that define the foliations at the various nodes are:

$$\omega_{02} = (-24xy - 80x - y^2 + 48y)dx + (36x^2 - xy - 32x)dy$$

$$\begin{aligned}\omega_{03} &= (-80x^2y^2 - 60xy^2 - \frac{25600}{27}xy + 80/9y)dx+\\ &+ (-80x^3y - 24x^2y - \frac{25600}{27}x^2 + \frac{560}{9}x - 1)dy\end{aligned}$$

$$\omega_{021} = (-24xy^2 - 80xy - y^2 + 48y)dx + (12x^2y - 80x^2 - 2xy + 16x)dy$$

$$\begin{aligned}\omega_{032} &= (-160x^3y^2 - \frac{381}{4}x^2y^2 - \frac{1053}{320}xy^2 - \frac{51200}{27}xy - \frac{1701}{81920}y^2 - \frac{80}{9}y)dx\\ &+ (-80x^4y - \frac{123}{4}x^3y - \frac{1971}{1280}x^2y - \frac{25600}{27}x^2 - \frac{1701}{81920}xy + 80/9x)dy\end{aligned}$$

$$\begin{aligned}\omega_{0212} &= (-12x^2y^2 - 3xy^2 - 160xy + \frac{2}{25}y^2 + 16y)dx+\\ &(12x^3y + \frac{14}{5}x^2y - 80x^2 + \frac{2}{25}xy - 16x)dy\end{aligned}$$

$$\begin{aligned}\omega_{0321} &= (-240x^3y^2 - 126x^2y^2 - \frac{6183}{1280}xy^2 - \frac{1701}{40960}y^2 - \frac{25600}{9}y)dx+\\ &+ (-80x^4y - \frac{123}{4}x^3y - \frac{1971}{1280}x^2y - \frac{1701}{81920}xy - \frac{25600}{27}x + \frac{80}{9})dy\end{aligned}$$

$$\omega_{02121} = (-\frac{1}{5}xy^2 + \frac{4}{2}5y^2 - 240y)dx + (12x^3y + \frac{14}{5}x^2y + \frac{2}{25}xy - 80x - 16)dy.$$

In order to test the performance of `resolutionf`, we computed the resolution trees, up to height 10, of 50 randomly generated foliations of degree 2 belonging to each one of the three families that are defined over U_z as follows. Let λ and μ be homogeneous polynomials of degree one, and let $a_1, \dots, a_4, c_1, c_2, c_3$ be constants. Then, for $f = \lambda\mu + a_1\lambda + a_2\mu + a_3$, we have

Family 1: $(c_1(\lambda + a_2)^2 + c_2 f)dx + f dy$,
Family 2: $(c_1 f + c_2(\lambda + a_2))dx + f dy$;

and for $f = (\lambda + a_1)^2 + a_2\mu + a_3$, we have

Family 3: $(c_1 f + c_2\lambda + c_3)dx + f dy$.

For the significance of these families see [2]. Note that the singularities of a generic foliation of any of these types belongs to the line at infinity. The results we obtained are summed up in Table 4.

Family	Total time	Total number of resolutions	Average number of nodes per second	Average number of nodes per resolution	Average number of resolutions per second	Average time per resolution
1	30s	72	2.40	1.44	1.64	0.60s
2	8s	158	19.75	3.16	6.25	0.16s
3	16s	348	21.75	6.96	3.12	0.32s

TABLE 4. 50 foliations were tested for each family

One should not forget that the data of Table 4 was not obtained from truly random forms. Indeed, although the forms have been randomly generated, they belong to rather simple families of very small degree. A truly random form of slightly bigger degree may easily stall the program at the root of the tree.

A more or less random test of the procedure `hasDicritical` proved more dificult, because we did not find in the literature a sufficiently nice family of foliations with dicritical singularities at various heights. Thus, we devised our own test, by considering the form

$$\omega_{c_1,c_2} = c_1 x^3 y dx + (y^3 - xy - c_2 x^4) dy.$$

D. Jordan showed in [5] that the only algebraic solution of $\omega_{1,1}$ is the line at infinity. Since the program had already shown that $\omega_{1,1}$ had a dicritical singularity at height 8, it seemed reasonable to turn to this family for our tests.

Height of the tree	Number of foliations
3	2
4	8
5	14
6	14
7	6
8	28
7	14
10	4

TABLE 5. The 90 forms with a dicritical singularity up to height 10.

We tested the 400 forms ω_{c_1,c_2} with integer parameters $(c_1, c_2) \in [-10, 10] \times [-10, 10]$. The program searched for dicritical singularities up to height 10, and kept a tally of how many of the forms had its first singularity at each height up to 10. In fact, only 90 of the 400 forms tested had a dicritical singularity within this range. Of course this does not mean that the other 310 forms are free from dicritical singularities, since we did not generate the whole resolution tree. The results are displayed in Table 5. The second column of row k of this table shows the number of forms, out of the 400, whose first dicritical singularities appears at height k. The table begins with height 3 because we listed only the heights with a nonzero number of dicritical singularities.

Finally, it should be observed that Singular does not allow the user to create new types, nor does it have a special type for differential forms. We got around this problem by writing a Singular library (called `forms.lib`) to handle the operations with one and two forms, that were defined as polynomials. For example, a 1-form in x and y corresponds to a polynomial in the variables x, y and dx, dy. This library, and also `resfolia.lib`, can be obtained from the URL `http://www.dcc.ufrj.br/~collier/folia.html`.

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