ON THE CLASSIFICATION OF SIMPLE QUADRATIC DERIVATIONS OVER THE AFFINE PLANE

S. C. COUTINHO

ABSTRACT. We classify simple derivations induced by unimodular rows of length 2 whose entries have degree 2, over the ring of complex polynomials in two variables. As part of the proof we give several new families of simple derivations over this ring.

1. INTRODUCTION

Let R be a commutative ring, and let d be a derivation of R. An ideal I of R is *stable under* d if $d(I) \subseteq I$. Of course, there are always two ideals stable under d, namely $\{0\}$ and R itself. If R does not have any nonzero proper ideals stable under d then it is called *d-simple*. In this case we also say that d is a *simple derivation* of R.

Simple derivations have played an important rôle in the construction of examples in noncommutative algebra ever since the 1930s, when Ore extensions were first introduced; see [8, Chapter 1]. More recently, the d-simplicity of the ring of polynomials has also been used to produce new examples of nonholonomic irreducible modules over the Weyl algebra; see [4], [7].

Unfortunately, examples of derivations with respect to which a given ring is d-simple have proved rather difficult to find. Even over a polynomial ring, only a few families, constructed in a more or less *ad hoc* way, are known at present.

However, one may approach the production of simple derivations, in a more systematic way, using unimodular rows as a starting point. This is specially effective over the ring $S = \mathbb{C}[x, y]$, which is the case we consider in this paper. In order to describe our main results in more detail we need some notation.

Let $\mathfrak{U}_n$ be the set of unimodular rows $(a, b) \in \mathbb{C}[x, y]$ with $\max\{\deg(a), \deg(b)\} = n$. Given $u = (a, b) \in \mathfrak{U}_n$ let

$$d_u = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}.$$

We call d_u the *derivation induced* by the unimodular row u. Since d_u must be nonsingular, it follows that it is a simple derivation if and only if, for all $f \in \mathbb{C}[x, y] \setminus \mathbb{C}$, we have that $d_u(f) \notin (f)$. Our main result is a classification theorem of the simple derivations induced by rows in $\mathfrak{U}_2$. This is accomplished by finding a

Date: February 29, 2008.

2000 *Mathematics Subject Classification.* Primary: 11R04, 37F75, 34M45; Secondary: 32S65.

Key words and phrases. derivation, singularity, algebraic solution.

I wish to thank D. Levcovitz, L. N. Bertoncello and the referee for their many useful comments and suggestions. During the preparation of this paper I was partially supported by grants from CNPq and Pronex (commutative algebra and algebraic geometry).

parametric representation for each family of elements in $\mathcal{U}_2$, and then embedding each of these parameterized sets in a corresponding affine variety.

It follows from Theorem 3.1 that there are five distinct types of parametric representations for the rows in $\mathcal{U}_2$. Each of these gives rise to a different variety $\mathcal{P}_j$, $1 \leq j \leq 5$, as explained above. The $\mathcal{P}_j$ will be called the spaces of rows of type j. However, as is usually the case, we pay a price for requiring $\mathcal{P}_j$ to be a closed set: not all rows in $\mathcal{P}_j$ are unimodular. However, $\mathcal{U}_2 \cap \mathcal{P}_j$ is dense in $\mathcal{P}_j$. For a precise definition of these parameter spaces see section 3. Let Δ_j be the set of unimodular rows $u \in \mathcal{P}_j$ for which d_u is a simple derivation.

The main theorem of the paper can now be stated as follows.

Theorem 1.1. *Let S_n be the subspace of S generated by the polynomials of degree less than or equal to n, and denote by $\overline{\mathcal{U}}_2$ the closure of $\mathcal{U}_2$ in $S_2 \times S_2$. Then $\mathcal{P}_1, \ldots, \mathcal{P}_4$ are the irreducible components of $\overline{\mathcal{U}}_2$. Moreover,*

1. $\dim(\mathcal{P}_j) = 8$, *for* $1 \leq j \leq 3$, *and* $\dim(\mathcal{P}_4) = 7$.
2. $\Delta_j \cap \mathcal{P}_j \neq \emptyset$ *for* $j = 1, 2, 3, 4$, *and*
3. Δ_j *is dense in* $\mathcal{P}_j$, *for* $j = 1, 2$.

Note that $\mathcal{P}_5$ is not an irreducible component of $\overline{\mathcal{U}}_2$. Indeed, as we show in Proposition 3.3, $\mathcal{P}_5 \subset \mathcal{P}_3$.

The paper is organized as follows. Section 2 contains a number of basic results from the theory of holomorphic foliations, that will be required in later sections. The parameterizations of the four types of unimodular rows of $\mathcal{U}_2$ are given in section 3, where we also construct the parameter spaces. Sections 4 through 6 contain the construction of examples of simple derivations corresponding to rows of types 1, 2 and 3; while the proof of Theorem 1.1 is given in the last section. It should be pointed out that for rows of types 1 and 2 we manage to construct families of derivations, but for types 3 and 4 only isolated examples are known. This explains why it has not been possible to prove Theorem 1.1(3) for $j = 3$ and $j = 4$. The obstacle to the construction of such families seems to be that the corresponding holomorphic foliations have dicritic singularities for rows of type 3 and degenerated singularities for those of type 4. Thus, we may pose the following problem.

Problem 1.2. *Is it true that Δ_j is dense in $\mathcal{P}_j$ for $j = 3$ and $j = 4$?*

Most of the results in this paper were first obtained by running experiments using the computer algebra systems Singular [10] and Axiom [6]. In the end it turned out to be possible to give complete proofs of all the results, with the exception of Theorem 1.1(3), without having to resort to a computer. However, the experiments were crucial in actually arriving at the results themselves.

2. Preliminaries

We begin with some basic results on holomorphic foliations over the complex projective plane $\mathbb{P}^2$; see [17] for more details.

Let $n \geq 1$ be an integer, and denote by x, y and z the homogeneous coordinates of the complex projective plane $\mathbb{P}^2$. A *holomorphic foliation* $\mathcal{F}$ of $\mathbb{P}^2$ is defined by a 1-form $\Omega = Adx + Bdy + Cdz$, where A, B and C are homogeneous polynomials of degree $n+1$ that satisfy the identity $xA + yB + zC = 0$. A 1-form that satisfies these conditions is said to be *projective*, and its *degree* is the integer n. A *singularity* of $\mathcal{F}$

is a point of $\mathbb{P}^2$ that is a common zero of A, B and C. Thus, the set of singularities of $\mathcal{F}$, which we denote by $\mathrm{Sing}(\mathcal{F})$ or $\mathrm{Sing}(\Omega)$, is a subset of $\mathbb{P}^2$. In this paper we deal only with foliations whose singular set is finite.

We say that a homogeneous polynomial $F \in \mathbb{C}[x, y, z]$ is an *algebraic solution* of Ω if there exists a 2-form η with coefficients in $\mathbb{C}[x, y, z]$ such that

$$\Omega \wedge dF = F\eta.$$

Proposition 2.1. *Let $\mathcal{F}$ be a foliation induced by a projective 1-form $\Omega = Adx + Bdy + Cdz$ with finite singular set. The following conditions are equivalent:*

1. *z is an algebraic solution of Ω;*
2. *$A(x, y, 0) = B(x, y, 0) = 0$;*
3. $\deg(\Omega) = \max\{\deg(A(x, y, 1)), \deg(B(x, y, 1))\}$.

Proof. We prove first the equivalence between (1) and (2). By definition, z is an algebraic solution of Ω if and only if

$$\Omega \wedge dz = Adxdz + Bdydz$$

can be written in the form $z\eta$ for some 2-form η. However, this happens if and only if both A and B are multiples of z; which is equivalent to saying that $A(x, y, 0) = B(x, y, 0) = 0$.

To prove the equivalence between (2) and (3), suppose that z^k divides both A and B. Thus, from $xA + yB + zC = 0$ we have that z^{k-1} divides C. Since we are assuming that the singularity set of $\mathcal{F}$ is finite, it follows that (2) is equivalent to $k = 1$. But, $k = 1$ if and only if

$$\deg(\Omega) = \max\{\deg(A(x, y, 1)), \deg(B(x, y, 1))\};$$

which completes the proof. □

Let U_z be the open set of $\mathbb{P}^2$ defined by $z \neq 0$ and let ω be the dehomogenization of Ω with respect to z. Restricting the foliation of $\mathbb{P}^2$ defined by Ω to U_z, we obtain the foliation of $\mathbb{C}^2$ defined by ω. If $\omega = bdx - ady$, write $d = a\partial/\partial x + b\partial/\partial y$. We say that ω is the *dual* 1*-form* of the derivation d. The foliation defined by Ω coincides in U_z with the foliation induced by the vector field corresponding to d.

Conversely, if $\pi_z : U_z \to \mathbb{C}^2$ is the map given by $\pi_z[x : y : z] = (x/z, y/z)$, then $\Omega = z^k \pi_z^*(\omega)$, where k is chosen so as to clear the poles of the pullback $\pi_z^*(\omega)$. If ω is the dual 1-form of a derivation d of $\mathbb{C}[x, y]$ we say that the resulting foliation $\mathcal{F}_d$ of $\mathbb{P}^2$ is *induced by* d. Since $\mathrm{Sing}(\mathcal{F}_d) \cap U_z$ coincides with the singular set of d, it follows that if d is a simple derivation then all the singularities of $\mathcal{F}_d$ belong to the line at infinity L_∞ with equation $z = 0$.

Throughout this section we assume that $\mathcal{F}_d$ is the foliation induced by the derivation

$$d = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y},$$

where $a, b \in \mathbb{C}[x, y]$.

Let p be a singularity of $\mathcal{F}_d$. If $p \in U_z$ and $d = a\partial/\partial x + b\partial/\partial y$, then $a(p) = b(p) = 0$. Let λ_1 and λ_2 be the eigenvalues of the jacobian matrix

$$J_p(d) = \begin{bmatrix} \partial a/\partial x & \partial a/\partial y \\ \partial b/\partial x & \partial b/\partial y \end{bmatrix}$$

Then p is a *reduced* singularity of $\mathcal{F}_d$ if $\lambda_2 \neq 0$ and either

- $\lambda_1 = 0$, or
- λ_1/λ_2 is not a positive rational number.

Similar definitions hold over the open sets U_y and U_x. Moreover, the concept is independent of the open set U in which the jacobian matrix is computed, because the ratio of eigenvalues remains unchanged when the derivation that defines $\mathcal{F}_d$ on U is multiplied by a holomorphic function that does not vanish on U.

Given a reduced (squarefree) nonconstant polynomial $f \in \mathbb{C}[x, y]$, we say that f is an *algebraic solution* of d if the homogenization of f with respect to z is an algebraic solution of $\mathcal{F}_d$. This is equivalent to saying that there exists $g \in \mathbb{C}[x, y]$ such that $d(f) = gf$. The polynomial g is called the *cofactor* of f. The derivation d is simple if and only if $\mathrm{Sing}(\mathcal{F}_d) \subset L_\infty$ and d has no algebraic solutions. The following two results will be the key to the construction of several examples in this paper.

Proposition 2.2. *If a derivation of* $\mathbb{Q}[i][x, y]$ *has an algebraic solution over* $\mathbb{C}[x, y]$*, then it has an algebraic solution with coefficients in* $\mathbb{Q}[i]$.

The proof is essentially the same as [12, proposition 3.3, p. 36] or [5, Proposition 2.1], so we do not give it here.

Theorem 2.3. *Assume that all the singularities of* $\mathcal{F}_d$ *are reduced and that* z *is an algebraic solution of* $\mathcal{F}_d$*. If* $f \in \mathbb{C}[x, y] \setminus \mathbb{C}$ *is an algebraic solution of* d*, then*

$$\deg(f) \leq m + 1,$$

where $m = \max\{\deg(a), \deg(b)\}$.

Proof. Let $F \in \mathbb{C}[x, y, z]$ be the homogenization of f with respect to z. Since f is reduced, so is F. Thus, the curves $F = 0$ and $z = 0$ are algebraic solutions of $\mathcal{F}_d$. Since L_∞ is not a component of $F = 0$, it follows that zF is also a reduced algebraic solution of $\mathcal{F}_d$. But

$$\deg(zF) = \deg(F) + 1 \leq \deg(\mathcal{F}_d) + 2$$

by [2] or [16, Theorem 2.3, p. 58]. However, since $\mathcal{F}_d$ has $z = 0$ as an algebraic solution, it follows from Proposition 2.1 that $\deg(\mathcal{F}_d) = m$. Hence,

$$\deg(f) = \deg(F) \leq m + 1,$$

as desired. □

3. The classification theorem

In this section we give a complete characterization of the unimodular rows of length 2 whose entries are polynomials of degree 2. In order to do that we introduce some notation. Let

$$G \cong GL_2(\mathbb{C}) \rtimes \mathbb{C}^2$$

be the complex affine group. Let f_k denote the homogeneous component of degree k of a polynomial $f \in \mathbb{C}[x, y]$. Assume that f is a quadratic polynomial; that is $f \in S_2$. Since f has complex coefficients, f_2 may be factored as a product of two linear homogeneous polynomials. Depending on whether these polynomials are linearly independent or not, we may write

$$f = \begin{cases} \lambda\mu + a_1\mu + a_2\lambda + a_3, \\ \qquad \text{or} \\ (\lambda + a_1)^2 + a_3\mu + a_4, \end{cases}$$

where λ and μ are linearly independent homogeneous polynomials and $a_1, \ldots, a_4 \in \mathbb{C}$. Thus, if $\gamma \in G$ is given by

$$\gamma(x, y) = (\lambda + a_1, \mu + a_2),$$

then,

$$(\gamma^{-1})^*(f) = \begin{cases} xy + (a_3 - a_1 a_2), \\ \qquad \text{or} \\ x^2 + a_3 y + (a_4 - a_2 a_3). \end{cases}$$

Defining a left action of $\gamma \in G$ into a unimodular row $u = (f, g)$ of $\mathcal{U}_2$ by

$$u \cdot \gamma = (\gamma^*(f), \gamma^*(g)),$$

we can prove the following classification result.

Theorem 3.1. *Let u be a row of $\mathcal{U}_2$ whose entries have degree 2. Then there exists $\gamma \in G$, such that*

$$u \cdot \gamma = (f, g) \quad \text{or} \quad u \cdot \gamma = (g, f)$$

where f, g is one of the pairs of polynomials in Table 1, under the corresponding nondegeneracy condition.

Type	f	g	*Nondegeneracy conditions*
1	$xy + a_1$	$a_2 f + a_3 x^2$	$a_1 a_2 a_3 \neq 0$
2	$xy + a_1$	$a_2 f + a_3$	$(a_3 - a_1 a_2) a_2 \neq 0$
3	$xy + a_1$	$a_2 f + a_3 x$	$a_1 a_2 a_3 \neq 0$
4	$x^2 + a_1 y + a_2$	$a_3 f + a_4$	$a_3 a_4 \neq 0$
5	$x^2 + a_2$	$a_3 x^2 + a_4 x + a_5$	$a_4 a_3 (a_2 a_4^2 + a_5^2) \neq 0$

TABLE 1. Canonical rows

Proof. From the comments preceding the statement of the theorem, we know that it is possible to assume that f is of one of two forms. This leads us to split the analysis in several different cases.

FIRST CASE: $f = xy + a_1$ and $g_2 \notin \mathbb{C}xy$.

Without loss of generality we may assume that $g_2(1, 0) \neq 0$. Denoting by a_2 the coefficient of xy in g, the system $f = g = 0$ is equivalent to

$$f = xy + a_1 = 0$$
$$h = g - a_2 f = (c_0 x + c_1)^2 + \nu(y) = 0,$$

where ν is a polynomial in y of degree less than or equal to 2. The condition $g_2(1, 0) \neq 0$ is equivalent to $c_0 \neq 0$. Eliminating x between f and h we find that the y-coordinates of the common zeroes of f and g are exactly the roots of the polynomial

$$p = (c_0 a_1 - c_1 y)^2 + y^2 \nu(y) = 0.$$

However, if $a_1 \neq 0$ then the constant term of p is nonzero, so the system $f = g = 0$ does not have zeroes if and only if all the other coefficients of p vanish. A simple calculation shows that this is equivalent to

$$c_1 = \nu = 0,$$

which gives rise to the polynomials

$$f = xy + a_1 \text{ and } g = c_0^2 x^2 + a_2(xy + a_1).$$

Taking $a_3 = c_0^2$, we have a row of type 1. We must still consider what happens when $a_1 = 0$. However, in this case, f vanishes whenever $y = 0$, and

$$g(x, 0) = (c_0 x + c_1)^2 + \nu(0) = 0$$

always has a solution because $c_0 \neq 0$. Hence, the system $f = g = 0$ always has a solution if $a_1 = 0$.

Second case: $f = xy + a_1$ and $g_2 \in \mathbb{C}xy$.

Let $g = c_1 xy + c_2 x + c_3 y + c_4$. Eliminating x between f and g we get the polynomial

$$p = c_3 y^2 + (-a_1 c_1 + c_4)y - a_1 c_2.$$

Thus, the system $f = g = 0$ does not have a solution if and only if, either p does not have any roots, or $a_1 \neq 0$ and its only root is $y = 0$.

In order that p be a nonzero constant polynomial, we must have that

$$c_3 = -a_1 c_1 + c_4 = 0 \text{ and } a_1 c_2 \neq 0.$$

Therefore,

$$g = c_1(xy + a_1) + c_2 x,$$

which gives the type 3 rows if we make $a_2 = c_1$ and $a_3 = c_2$.

Now, if $a_1 \neq 0$ then $y = 0$ is not a zero of $xy + a_1$, so the system does not have a solution if 0 is the only root of p. But this happens when

$$c_3 \neq 0 \text{ and } -a_1 c_1 + c_4 = c_2 = 0,$$

or when

$$-a_1 c_1 + c_4 \neq 0 \text{ and } c_3 = c_2 = 0.$$

The former condition also corresponds to rows of type 3 (if we swap x with y in the canonical row), while the second gives

$$g = c_1 xy + c_4,$$

which corresponds to rows of type 2 if we make $a_2 = c_1$ and $a_3 = c_4$. On the other hand, if $a_1 = 0$ then either $x = 0$ or $y = 0$, and g always has a solution in this case.

This settles all the cases in which at least one of the projective curves defined by the homogenizations of f or g has distinct points of intersection with L_∞. Therefore, we may now suppose that $f = x^2 + a_1 y + a_2$.

Third case: $f = x^2 + a_1 y + a_2$.

In this case $g_2 \in \mathbb{C}x^2$ for, otherwise, f and g would intersect outside L_∞. Thus we can write $g = c_1 f + c_2 x + c_3 y + c_4$. If $c_2 = c_3 = 0$ we end up with a row of type 4; while if $c_2 = 0$ but $c_3 \neq 0$, the row is not unimodular. Hence, we may assume, from now on, that $c_2 \neq 0$. Eliminating x between f and

$$h = g - c_1 f = c_2 x + c_3 y + c_4,$$

we find that the resultant of f and g with respect to x is

$$p = c_3^2 y^2 + (a_1 c_2^2 + 2c_3 c_4)y + (a_2 c_2^2 + c_4^2).$$

Proceeding as above there are three possibilities if (f, g) is to be a unimodular row. The first is that

$$a_1c_2^2 + 2c_3c_4 = c_3 = 0 \text{ and } \quad a_2c_2^2 + c_4^2 \neq 0,$$

which is equivalent to

$$c_3 = a_1 = 0 \text{ and } \quad a_2c_2^2 + c_4^2 \neq 0.$$

Thus,

$$f = x^2 + a_2$$
$$g = c_1 f + c_2 x + c_4,$$

whenever $a_2c_2^2 + c_4^2 \neq 0$. Taking $a_3 = c_1$, $a_4 = c_2$ and $a_5 = c_4$, we end up with a row of type 5.

The second case occurs when

$$a_1c_2^2 + 2c_3c_4 = a_2c_2^2 + c_4^2 = 0, \text{ and } \quad c_3 \neq 0.$$

Since $c_2 \neq 0$ we get that $a_2 = -c_4^2/c_2^2$ and $a_1 = -2c_3c_4/c_2^2$, and the resulting system is

$$c_2^2 f = c_2^2 x^2 - c_4(2c_3 y + c_4) \text{ and } \quad g = c_1 f + c_2 x + c_3 y + c_4.$$

However,

$$c_2^2 f(x, 0) = c_2^2 x^2 - c_4^2 = (g(x, 0) - c_1 f(x, 0))(c_2 x - c_4),$$

so this system always has the solution $x = -c_4/c_2$ and $y = 0$. Finally,

$$c_3 = a_2c_2^2 + c_4^2 = 0,$$

and $a_1c_2^2 + 2c_3c_4 \neq 0$, which gives

$$f = x^2 + a_1 y + a_2 \text{ and } \quad g = c_1 f + c_2 x + c_4.$$

However, these polynomials always have a common zero because $c_2 \neq 0$. $\square$

For each $1 \leq j \leq 5$, we define a *canonical* row of type j to be a vector (f, g), with f and g as given in row j of Table 1, *but with no restriction on their coefficients.* In other words, we are disregarding the nondegeneracy conditions for the sake of this definition. Thus, the set $\mathcal{C}_j$ of all canonical rows of type j is isomorphic to an affine space over $\mathbb{C}$. The dimension of $\mathcal{C}_j$ is 3 for $1 \leq j \leq 3$, and 4 for $j = 4$. Since, $\mathcal{U}_2$ is invariant under the action of G, it follows that the set of unimodular rows of type j is contained in $\mathcal{C}_j \cdot G$. Let $\mathcal{P}_j$ denote the closure of $\mathcal{C}_j \cdot G$ in S_2^2.

Corollary 3.2. *If $(f, g) \in \mathcal{P}_j$, then $(g, f) \in \mathcal{P}_j$, for all $1 \leq j \leq 4$.*

Proof. By the definition of $\mathcal{P}_j$, we may restrict the proof to a Zariski dense subset of $\mathcal{C}_j$. For type 1, we have that if $f = xy + a_1$ and $g = a_2 f + a_3 x^2$, with $a_2 \neq 0$, then

$$g = x(a_2 y + a_3 x) + a_1 a_2.$$

Changing coordinates to x and $y' = a_2 y + a_3 x$, we have that $g = xy' + a_1 a_2$ and

$$f = \frac{1}{a_2} g - \frac{a_3}{a_2} x^2$$

which is of type 1. If (f, g) is a canonical row of type 3, then $f = xy + a_1$ and $g = a_2 f + a_3 x$. Writing,

$$g = x(a_2 y + a_3) + a_1 a_2,$$

and taking $y' = a_2y + a_3$, we find that if $a_2 \neq 0$, then

$$f = \frac{1}{a_2}g - \frac{a_3}{a_2}x;$$

so $(g, f) \in \mathcal{P}_2$. The other two cases are even simpler, and will be left to the reader. $\square$

The first step in determining the irreducible components of $\overline{\mathcal{U}}_2$ is to dispose of the rows of type 5.

Proposition 3.3. *If u is a unimodular row of type 5, then $u \in \mathcal{P}_3$. Moreover, d_u has an algebraic solution for every generic $u \in \mathcal{P}_5$.*

Proof. We begin by showing that a unimodular row of type 5 is a degenerate case of a type 3 row. Let

$$u = (x^2 + a_2, a_3x^2 + a_4x + a_5) \in \mathcal{C}_5.$$

For

$$h = \frac{a_4^2xy + a_5^2 - 2a_2a_3a_5 + a_2a_4^2 + a_2^2a_3^2}{a_4^2},$$

consider the type 3 row

$$v = (h, a_3h + x).$$

A straightforward calculation shows that if

$$\gamma = \frac{1}{a_4^2}\left(a_4^2(a_4x + a_5 - a_2a_3), \epsilon a_4^2y + a_4x - a_5 + a_2a_3\right)$$

then

$$u = \lim_{\epsilon \to 0} v \cdot \gamma.$$

Since $\mathcal{P}_3$ is closed under the Zariski topology, it is also closed under the analytic topology. Hence, $u \in \mathcal{P}_3$, which implies that $\mathcal{C}_5 \subset \mathcal{P}_3$. Thus, $\mathcal{P}_5 \subset \mathcal{P}_3$, as had been claimed.

Next we prove that a derivation corresponding to a row of type 5 must have a solution. A general unimodular row u of type 5 can be written in the form

$$u = (\lambda^2 + b_1\lambda + b_0, c_2\lambda^2 + c_1\lambda + c_0)$$

where $\lambda = a_1x + a_2y$. Thus,

$$d_u = (\lambda^2 + b_1\lambda + b_0)\frac{\partial}{\partial x} + (c_2\lambda^2 + c_1\lambda + c_0)\frac{\partial}{\partial y},$$

Let

$$k = a_1 + c_2a_2, \quad \ell = b_1a_1 + c_1a_2 \text{ and } m = b_0a_1 + c_0a_2.$$

If u belongs to the open set of $\mathcal{P}_5$ defined by $a_1a_2km \neq 0$, then $h = \lambda^2 + e_1\lambda + e_0$ is an algebraic solution of d_u with co-factor $g = 2k\lambda + v$, for all e_1, e_2, v that satisfy the equations

$$-v - ke_1 + 2\ell = -e_1v - 2ke_0 + e_1\ell + 2m = e_1m - ve_0 = 0.$$

As one readily checks, this set of equations always has a solution. $\square$

The next proposition gives the first part of Theorem 1.1.

Proposition 3.4. *$\mathcal{P}_1, \ldots, \mathcal{P}_4$ are the irreducible components of $\overline{\mathcal{U}}_2$. Moreover,*

$$\dim(\mathcal{P}_j) = 8, \textit{ for } 1 \leq j \leq 3 \textit{ and } \dim(\mathcal{P}_4) = 7.$$

Proof. Now let $\mathring{\mathcal{U}}_2$ be the set of rows of $\mathcal{U}_2$ whose entries have, both of them, degree 2. Then, by Theorem 3.1 and Proposition 3.3,

$$\mathring{\mathcal{U}}_2 \subseteq \bigcup_{j=1}^{4} \mathcal{P}_j \subseteq \overline{\mathcal{U}_2}.$$

However, $\mathring{\mathcal{U}}_2$ is a dense subset of $\mathcal{U}_2$. Thus, taking the Zariski closures of the three sets above, we find that

$$\bigcup_{j=1}^{4} \mathcal{P}_j = \overline{\mathcal{U}_2}.$$

In order to prove that the sets $\mathcal{P}_j$ are irreducible, consider the morphism

$$\Phi : G \times \mathcal{C}_j \to \mathcal{P}_j,$$

given by $\Phi(\gamma, u) = u \cdot \gamma$. Since

$$\overline{\Phi(\mathcal{C}_j)} = \mathcal{P}_j,$$

and $G \times \mathcal{C}_j$ is irreducible, the set $\mathcal{P}_j$ is also irreducible by [13, Proposition 1, §8, p. 48]. In particular, these are the irreducible components of $\overline{\mathcal{U}}_2$.

Now, consider the morphism

$$\Phi : G \times \mathcal{C}_j \to \mathcal{P}_j,$$

given by $\Phi(\gamma, u) = u \cdot \gamma$. Since Φ is dominant, it follows that the closure of $\mathrm{Im}(\Phi)$ equals $\mathcal{P}_j$.

In order to find the dimension of $\mathcal{P}_j$ we need only compute the dimension of the generic fibre of Φ, which is equal to the dimension of the stabilizer

$$\mathrm{Stb}_G(u_j, \mathcal{C}_j) = \{\gamma \in G : u_j \cdot \gamma \in \mathcal{C}_j\} \subseteq G$$

of a generic element $u_j \in \mathcal{C}_j$. Thus,

$$\dim(\mathcal{P}_j) = 6 + \dim(\mathcal{C}_j) - \dim(\mathrm{Stb}_G(u_j, \mathcal{C}_j)),$$

because $\dim(G) = 6$. To finish the proof we must compute $\dim(\mathrm{Stb}_G(u_j, \mathcal{C}_j))$ for $1 \le j \le 4$.

Suppose, first, that $1 \le j \le 3$, and assume that $(xy + b_1) \cdot \gamma$ can be written in the form $xy + b_2$, where $\gamma \in G$ and b_1 and b_2 are nonzero complex numbers. This implies that γ is either a diagonal or an anti-diagonal matrix of $\mathrm{GL}_2(\mathbb{C})$ whose nonzero elements are α and α^{-1}, where α is a nonzero complex number. Since the argument is the same in both cases, we will assume that $\gamma = \mathrm{diag}(\alpha, \alpha^{-1})$. Note also that $b_1 = b_2$. We must now analyse the second coordinate of a canonical row for each of the types 1, 2, and 3.

However, $\gamma^*(g)$ is always of the same type as g, as shown in Table 2.

Type	g	$\gamma^*(g)$
1	$a_3x^2 + a_2(xy + b_1)$	$a_3\alpha^2x^2 + a_2(xy + b_1)$
2	$a_2(xy + b_1) + a_3$	$a_2(xy + b_1) + a_3$
3	$a_2(xy + b_1) + a_3x$	$a_2(xy + b_1) + a_3\alpha x$

TABLE 2. $\gamma^*(g)$ by row type

This implies that the dimension of the fibre over the generic row for any of these three types is 1. Moreover, for all the three types the dimension of $\mathcal{C}_j$ is 3. Therefore, $\dim(\mathcal{P}_j) = 8$, for $1 \leq j \leq 3$.

Finally, we consider rows of type 4. In this case the calculations are more involved. Assume that

$$\gamma = (q_1x + q_2y + q_5, q_3x + q_4y + q_6),$$

and that $f = x^2 + a_1y + a_2$. The coefficients of y^2 and x in $\gamma^*(f)$ are, respectively, q_2^2 and $2q_1q_5 + a_1q_3$. Therefore, if $\gamma^*(f)$ is to be of type 4, we must have

$$q_2 = 2q_1q_5 + a_1q_3 = 0 \text{ and } \quad q_1 = \pm 1, \tag{3.1}$$

Assuming, as we may, that $q_1 = 1$, we get

$$\gamma^*(f) = x^2 + a_1q_4y + (a_1q_6 + q_5^2 + a_2).$$

Moreover, since $g = a_3f + a_4$, it follows that

$$\gamma^*(g) = a_3\gamma^*(f) + a_4,$$

which is of the required type. Therefore, the fibre over a generic row of type 4 has dimension 3. Since $\dim(\mathcal{C}_4) = 4$, it follows that $\dim(\mathcal{P}_4) = 7$. $\square$

We end the section with a description of the strategy that will be used to prove that the set Δ_j (of rows that correspond to derivations without solutions) is dense in $\mathcal{P}_j$, for $1 \leq j \leq 2$. We begin by defining a new left action of $G = \mathrm{GL}_2(\mathbb{C}) \rtimes \mathbb{C}^2$ on S_2^2. Given a row $u = (f, g)$, let $\omega_u = gdx - fdy$, which is the 1-form dual to the derivation d_u. Now, if

$$\gamma = (q_1x + q_2y + q_5, q_3x + q_4y + q_6),$$

is an element of G, then

$$\gamma^*(\omega_u) = (\gamma^*(g)q_1 - \gamma^*(f)q_3)dx + (\gamma^*(g)q_2 - \gamma^*(f)q_4)dy.$$

Thus, we define a left action of G in U_2 by

$$(f, g) \circ \gamma = (\gamma^*(g)q_2 - (\gamma^*(f)q_4), -\gamma^*(g)q_1 + \gamma^*(f)q_3).$$

Since $\mathcal{U}_2$ is invariant under the action $\circ$, then so is $\overline{\mathcal{U}}_2$. But G is a connected algebraic group so, from [9, section 8.2, Proposition, p. 59], each irreducible component of $\overline{\mathcal{U}}_2$ is invariant under G. Thus, if $\mathcal{A}_j$ is a subset of $\mathcal{P}_j$ then $G \cdot \mathcal{A}_j \subseteq \mathcal{P}_j$. As in the proof of Proposition 3.4, we define

$$\Psi_j : G \times \mathcal{A}_j \to \mathcal{P}_j,$$

given by $\Psi_j(\gamma, u) = u \circ \gamma$.

Proposition 3.5. *Let $1 \leq j \leq 2$, and assume that the closure of $\mathrm{Im}(\Psi_j)$ has dimension 8. If $\Delta_j \cap \mathcal{A}_j$ is dense in $\mathcal{A}_j$, then Δ_j is dense in $\mathcal{P}_j$.*

Proof. By Proposition 3.4 and the hypothesis on the dimension of the closure of $\mathrm{Im}(\Psi_j)$, we have that

$$\overline{\mathcal{A}_j \circ G} = \mathcal{P}_j. \tag{3.2}$$

Since $\Delta_j \cap \mathcal{A}_j$ is dense in $\mathcal{A}_j$, it follows from (3.2) that $(\Delta_j \cap \mathcal{A}_j) \circ G$ is dense in $\mathcal{P}_j$. However, the derivation that corresponds to a row $u \circ \gamma$, with $u \in \Delta_j$, has

$$\omega_{u\circ\gamma} = \gamma^*(\omega_u)$$

as its dual 1-form. This implies that $d_{u\circ\gamma}$ is simple, and completes the proof. $\square$

Thus, in order to prove (3) of Theorem 1.1 it is enough to (1) analyse each type of derivation separately, looking for large enough families of explicit examples, and (2) compute the dimension of $\mathrm{Im}(\Psi_j)$. The construction of the families is quite elaborate and will be carried out in sections 4 through 6; while $\dim(\mathrm{Im}(\Psi_j))$ will be determined in section 7 using a computer algebra system.

4. Type 1

In this section we show that a generic derivation of type 1 over the gaussian rationals $\mathbb{Q}[i]$ is simple.

Proposition 4.1. *Let $a \neq 1$, b and c be nonzero rational gaussian numbers such that the polynomial $y^2 + bx^2 + cxy$ is irreducible over $\mathbb{Q}[i]$. Then, the derivation*

$$d = (xy + a)\frac{\partial}{\partial x} - (c(xy + a) + bx^2)\frac{\partial}{\partial y}$$

is simple in $\mathbb{C}[x, y]$.

Proof. Let $\mathcal{F}$ be the foliation of $\mathbb{P}^2$ induced by d. The proof consists in showing that all the singularities of $\mathcal{F}$ are reduced, and then using Theorem 2.3 to bound the degree of all possible algebraic solutions of d. We can then determine what these solutions are.

In the open set U_z the foliation $\mathcal{F}$ is defined by the 1-form $(c(xy+a)+bx^2)dx+(xy + a)dy$ whose homogenization with respect to z is

$$\Omega = z(cxy + caz^2 + bx^2)dx + z(xy + az^2)dy - (cx^2y + caxz^2 + bx^3 + xy^2 + ayz^2)dz.$$

The singularities of Ω are readily seen to be $[0 : 1 : 0]$ and the points $[x_0 : y_0 : 0]$ such that $y_0^2 + bx_0^2 + cx_0y_0 = 0$. In particular, all the singularities of Ω belong to the open set U_y. Note that since $bc \neq 0$, it follows that $x_0y_0 \neq 0$. Dehomogenizing Ω at y, we find that $\mathcal{F}$ is defined on U_y by the vector field

$$(cx^2 + caxz^2 + bx^3 + x + az^2)\frac{\partial}{\partial x} + z(cx + caz^2 + bx^2)\frac{\partial}{\partial z},$$

whose jacobian at a point on the line L_∞ is

$$J(x, y) = \begin{bmatrix} 2cx + 3bx^2 + 1 & 0 \\ 0 & cx + bx^2 \end{bmatrix}$$

Therefore, the eigenvalues of $J(0, 0)$ are 0 and 1, and the foliation is reduced at this point. On the other hand, the eigenvalues of $J(x_0, y_0)$ are $2cx_0 + 3bx_0^2 + 1$ and $cx_0 + bx_0^2$ both of which are nonzero, because $bx_0^2 + cx_0 + 1$ is irreducible over $\mathbb{Q}[i]$. Moreover, the ratio

$$\frac{2cx_0 + 3bx_0^2 + 1}{cx_0 + bx_0^2} = 3 - \frac{cx_0 - 1}{cx_0 + bx_0^2}$$

is rational if and only if $(cx_0 - 1)/(cx_0 + bx_0^2) = q \in \mathbb{Q}$. But this implies that

$$qbx_0^2 + c(q - 1)x_0 + 1 = 0,$$

which cannot happen for any $q \in \mathbb{Q}$ because $bx^2 + cx + 1$ is irreducible over $\mathbb{Q}[i]$. Thus, it follows from Proposition 2.2 and Theorem 2.3 that if d is not simple then it must have an algebraic solution with coefficients in $\mathbb{Q}[i]$ and degree at most 3.

Assume, by contradiction, that d has an algebraic solution $f \in \mathbb{Q}[i][x, y]$ of degree $k \leq 3$. Thus, there exists $g \in \mathbb{Q}[i][x, y]$ such that

$$d(f) = gf \tag{4.1}$$

and $\deg(g) \leq 1$. Equating the homogeneous components of degree $j+1$ on both sides of (4.1), we find that

$$xy\frac{\partial f_j}{\partial x} + a\frac{\partial f_{j+2}}{\partial x} - (cxy + bx^2)\frac{\partial f_j}{\partial y} - ca\frac{\partial f_{j+2}}{\partial y} = g_1 f_j + g_0 f_{j+1}.$$

Applying Euler's relation

$$x\partial f_j/\partial x + y\partial f_j/\partial y = jf_j,$$

to the first term, we end up with

$$h\frac{\partial f_j}{\partial y} - a\frac{\partial f_{j+2}}{\partial x} + ca\frac{\partial f_{j+2}}{\partial y} = (jy - g_1)f_j - g_0 f_{j+1}, \tag{4.2}$$

where $h = y^2 + cxy + bx^2$. In particular, if $j = k$, we get that

$$h\frac{\partial f_k}{\partial y} = (ky - g_1)f_k. \tag{4.3}$$

Since h is irreducible over $\mathbb{Q}[i]$ by hypothesis, it follows from (4.3) that either $k \geq 2$ and h divides f_k, or

$$g_1 = ky \quad \text{and} \quad f_k = x^k,$$

where, without loss of generality, we have assumed the nonzero coefficient of x^k to be one.

Let us consider first the case where $f_k = \lambda h$, for some nonzero homogeneous polynomial $\lambda \in \mathbb{Q}[i][x,y]$ of degree at most 1. Taking this into (4.3) and cancelling h throughout the equation we get that

$$\lambda\frac{\partial h}{\partial y} + h\frac{\partial \lambda}{\partial y} = (ky - g_1)\lambda.$$

Since λ does not divide h, this implies that $\partial\lambda/\partial y = 0$. Hence,

$$g_1 = ky - \frac{\partial h}{\partial y} = (k-2)y - cx.$$

Taking this into (4.2) with $j = k-1$, we get

$$h\frac{\partial f_{k-1}}{\partial y} = (y + cx)f_{k-1} - g_0\lambda h.$$

Therefore, $f_{k-1} = \alpha h$, for some $\alpha \in \mathbb{C}$. Together with the previous equation, this implies that

$$\alpha\frac{\partial h}{\partial y} = \alpha(y + cx) - g_0\lambda.$$

Hence, $\alpha y = -g_0\lambda$. Since $\partial\lambda/\partial y = 0$, we must have that $\alpha = g_0 = f_{k-1} = 0$.

Taking all this into (4.2) with $j = k-2$, it follows that

$$h\frac{\partial f_{k-2}}{\partial y} - a\left(h\partial\lambda/\partial x - \lambda(cy + (c^2 - 2b)x)\right) = cxf_{k-2}. \tag{4.4}$$

We now consider two cases depending on whether $k = 2$ or $k = 3$. In the former case, λ and f_{k-2} must be constant and equation (4.4) becomes

$$a\lambda(cy + (c^2 - 2b)x) = cxf_0,$$

which is a contradiction because $ac\lambda \neq 0$ by hypothesis. On the other hand, if $k = 3$ then (4.4) becomes

$$h\frac{\partial f_1}{\partial y} - a\left(h\frac{\partial \lambda}{\partial x} - \lambda(cy + (c^2 - 2b)x)\right) = cxf_1.$$

Since $\partial\lambda/\partial y = 0$, we may assume, without loss of generality, that $\lambda = x$, so that

$$h\left(\frac{\partial f_1}{\partial y} - a\right) = x(cf_1 - a(cy + (c^2 - 2b)x)).$$

But h is irreducible over $\mathbb{Q}[i][x, y]$, which implies that

$$\partial f_1/\partial y = a \ \text{ and } \ cf_1 = a(cy + (c^2 - 2b)x).$$

Taking this into

$$-a\frac{\partial f_1}{\partial x} + ca\frac{\partial f_1}{\partial y} = 0;$$

which is (4.2) with $j = -1$, we get

$$-\frac{a}{c}(c^2 - 2b) + ca^2 = 0,$$

from which $b = 0$, which is a contradiction. This settles the first case.

Let us assume now that $g_1 = ky$ and that $f_k = x^k$, with $k \leq 3$. We will discuss only the case $k = 3$; the other two cases can be similarly handled. Take f and g to be polynomials with undetermined coefficients of the form

$$f = x^3 + u_1x^2 + u_2xy + u_3y^2 + u_4x + u_5y + u_6 \text{ and } \quad g = 3y + v.$$

Comparing the coefficients of y^3, xy^2, y^2, x^3 and x^2, respectively, in $d(f) = gf$ we get the system

$$\begin{aligned} -3u_3 &= 0 \\ -2cu_3 - 2u_2 &= 0, \\ -u_3v - 3u_5 &= 0, \\ -v - bu_2 &= 0, \\ -u_1v - bu_5 + 3a &= 0, \end{aligned}$$

From the first four equations it follows that $u_3 = u_2 = u_5 = v = 0$. Substituting this into the fifth equation we end up with $a = 0$, which is a contradiction. Thus, d does not have any algebraic solution, and the proposition is proved. □

5. Type 2

We now turn to derivations induced by type 2 unimodular rows. We begin with an explicit family of examples. However, this time, it is not restricted to degree 2. Let $\beta \in \mathbb{Q}[i][x, y]$ be a homogeneous irreducible polynomial of degree $n > 1$, and write

$$d = \frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}.$$

First of all, it follows from Proposition 2.2 that if d has an algebraic solution, then it has an algebraic solution with coefficients in the field of Gaussian numbers. Assuming that $f \in \mathbb{Q}[i][x,y]$ is an algebraic solution of d of degree k, we have that

$$\frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} = gf, \tag{5.1}$$

where $g \in \mathbb{Q}[i][x,y]$ has degree less than or equal to $n-1$. Moreover, since f is not constant it follows that $g \neq 0$.

Lemma 5.1. *There exists $m \geq 1$ such that*

$$f_k = x^{k-mn}\beta^m \quad \text{and} \quad g_{n-1} = m\partial\beta/\partial y.$$

In particular, $k \geq nm > m$.

Proof. Let g_ℓ be the nonzero homogeneous component of highest degree of g. Then, $g_\ell f_k \neq 0$, so that $\deg(g_\ell f_k) \geq k$. Hence, equating the homogeneous components of degree $\ell + k$ in (5.1), we get

$$\beta \frac{\partial f_{k+\ell-n+1}}{\partial y} = g_\ell f_k.$$

However, $\deg(g_\ell) < n = \deg(\beta)$. Since β is irreducible over $\mathbb{Q}[i]$, it must divide f_k; so that $\partial f_k/\partial y \neq 0$. Thus, equating the homogeneous components of degree $n+k-1$ on both sides of (5.1), we have that

$$0 \neq \beta \frac{\partial f_k}{\partial y} = g_{n-1} f_k. \tag{5.2}$$

In particular, $g_{n-1} \neq 0$ and $f_k = \beta^m h$, where h is homogeneous of degree $k - mn$. Note that we can also assume that $\gcd(h, \beta) = 1$, because β is irreducible over $\mathbb{Q}[i]$. Taking this into (5.2), and cancelling β^m throughout the resulting equation, we obtain

$$\beta \frac{\partial h}{\partial y} = h\left(g_{n-1} - m\frac{\partial \beta}{\partial y}\right).$$

Note that the terms in brackets have degree $n-1$. Since β is irreducible of degree n and does not divide h, we must have that

$$g_{n-1} = m\frac{\partial \beta}{\partial y} \text{ and that } \frac{\partial h}{\partial y} = 0.$$

Hence $h = cx^{k-mn}$, and the proof is complete. □

In fact, as we show in the next proposition, all the other homogeneous components of g must vanish.

Proposition 5.2. *We have that $g_{n-2} = \cdots = g_0 = 0$. In particular, $g = g_{n-1} = m\partial\beta/\partial y$.*

Proof. Assume, by induction on ℓ, that

- β^m divides f_{k-i},
- $g_{n-i-1} = 0$.

for all $i < \ell$. If $0 < \ell < n$, then

$$\deg\left(\beta \frac{\partial f_{k-\ell}}{\partial y}\right) = n + k - \ell - 1 = (n - \ell) + k - 1 > k - 1.$$

Thus, equating the homogeneous components of degree $n+k-\ell-1$ in (5.1), we get

$$\beta\frac{\partial f_{k-\ell}}{\partial y}=\sum_{j=0}^{\ell} g_{n-1-j}f_{k-\ell+j}. \tag{5.3}$$

Hence, by lemma 5.1, and the induction hypothesis,

$$\beta\frac{\partial f_{k-\ell}}{\partial y}=m\frac{\partial\beta}{\partial y}f_{k-\ell}+g_{n-\ell-1}x^{k-mn}\beta^m. \tag{5.4}$$

Therefore, β divides $f_{k-\ell}$. Let $f_{k-\ell}=\beta^s h$, where h is homogeneous of degree $k-\ell-ns$ and $\gcd(\beta,h)=1$. Taking this into (5.4), we have that

$$\beta\left(s\beta^{s-1}h\frac{\partial\beta}{\partial y}+\beta^s\frac{\partial h}{\partial y}\right)=m\frac{\partial\beta}{\partial y}\beta^s h+g_{n-\ell-1}x^{k-mn}\beta^m.$$

Note that if $g_{n-\ell-1}\neq 0$, then β^s must divide β^m, so that $s\leq m$. Cancelling now β^s throughout the equation, we obtain

$$\beta\frac{\partial h}{\partial y}-g_{n-\ell-1}x^{k-mn}\beta^{m-s}=(m-s)\frac{\partial\beta}{\partial y}h.$$

Since β does not divide $h\partial\beta/\partial y$, this can only happen if $m=s$. But this implies that

$$\beta\frac{\partial h}{\partial y}=g_{n-\ell-1}x^{k-mn},$$

so that $g_{n-\ell-1}=0$, and the proof is complete. □

Before we proceed to the proof of the main proposition of this section we need a technical result about homogeneous polynomials in two variables.

Lemma 5.3. *Let $\beta\in\mathbb{Q}[i][x,y]$ be a homogeneous polynomial of degree $n>1$. If*

$$\frac{\partial\beta}{\partial x}=c\frac{\partial\beta}{\partial y},$$

for some $c\in\mathbb{Q}[i]$, then β is reducible in $\mathbb{Q}[i][x,y]$.

Proof. Let

$$h(t)=\beta(t,1)=a_nt^n+\cdots+a_1t+a_0.$$

Then, $\beta(x,y)=y^nh(x/y)$. Hence,

$$\frac{\partial\beta}{\partial x}=y^{n-1}h'(\frac{x}{y})\ \text{ and }\ \frac{\partial\beta}{\partial y}=y^{n-1}(nh(\frac{x}{y})-\frac{x}{y}h'(\frac{x}{y})).$$

Writing t for x/y, the equality $\partial\beta/\partial x=c\partial\beta/\partial y$ becomes

$$h'(t)=c(nh(t)-th'(t)).$$

This gives rise to the recurrence

$$(n-j)\cdot c\cdot a_j=(j+1)a_{j+1},$$

which can be solved to give

$$a_j=c^j\binom{n}{j}a_0.$$

Therefore, $h(t)=a_0(ct+1)^n$, which implies that β is completely reducible over $\mathbb{Q}[i]$. □

Proposition 5.4. *Let $\beta \in \mathbb{Q}[i][x,y]$ be a homogeneous polynomial of degree $n \geq 2$, that is irreducible over $\mathbb{Q}[i]$. Then $\mathbb{C}[x,y]$ is d-simple with respect to $d = \partial/\partial x + \beta\partial/\partial y$.*

Proof. Let $f \in \mathbb{C}[x,y]$ be a nonconstant polynomial of degree $k > 0$ that satisfies $d(f) = gf$, for some $g \in \mathbb{C}[x,y]$. By Proposition 2.2 we may assume that both f and g have coefficients in $\mathbb{Q}[i]$.

Comparing homogeneous components of degree $k-1$, and taking Proposition 5.2 into account, we have that

$$\frac{\partial f_k}{\partial x} + \beta\frac{\partial f_{k-n}}{\partial y} = g_{n-1}f_{k-n}.$$

It then follows by Lemma 5.1 that

$$(k-mn)x^{k-mn-1}\beta^m + m\beta^{m-1}x^{k-mn}\frac{\partial \beta}{\partial x} + \beta\frac{\partial f_{k-n}}{\partial y} = mf_{k-n}\frac{\partial \beta}{\partial y}.$$

First case: $k > mn$.

Since $k - mn > 0$, it follows, by Euler's relation $x\partial\beta/\partial x = n\beta - y\partial\beta/\partial y$, that

$$kx^{k-mn-1}\beta^m + \beta\frac{\partial f_{k-n}}{\partial y} = (mf_{k-n} + m\beta^{m-1}x^{k-mn-1}y)\frac{\partial \beta}{\partial y}. \tag{5.5}$$

There are now two subcases, depending on whether $m = 1$ or $m > 1$.

Assume first that $m > 1$. It then follows from equation (5.5) that β divides f_{k-n}. Let $f_{k-n} = \beta^s h$, where $\gcd(\beta, h) = 1$. Taking this into (5.5), we get

$$kx^{k-mn-1}\beta^m + \beta^{s+1}\frac{\partial h}{\partial y} = \left((m-s)\beta^s h + m\beta^{m-1}x^{k-mn-1}y\right)\frac{\partial \beta}{\partial y}. \tag{5.6}$$

In particular, β^s divides

$$\beta^{m-1}\left(mx^{k-mn-1}y\frac{\partial \beta}{\partial y} - kx^{k-mn-1}\beta\right).$$

Since β does not divide the expression in brackets, it follows that $s \leq m-1$. This allows us to cancel β^s throughout (5.6), so that

$$kx^{k-mn-1}\beta^{m-s} + \beta\frac{\partial h}{\partial y} = \left((m-s)h + m\beta^{m-1-s}x^{k-mn-1}y\right)\frac{\partial \beta}{\partial y}.$$

Since β does not divide $h\partial\beta/\partial y$, this implies that $s = m-1$, and that

$$\beta\left(kx^{k-mn-1} + \frac{\partial h}{\partial y}\right) = \left(h + mx^{k-mn-1}y\right)\frac{\partial \beta}{\partial y}. \tag{5.7}$$

But this can only happen if $h = -mx^{k-mn-1}y + \alpha\beta$, where α is either zero or a nonzero homogeneous polynomial of degree $k-(m+1)n$. Taking this into (5.7) and cancelling β throughout the equation, we obtain

$$(k-m)x^{k-mn-1} + \left(\alpha\frac{\partial \beta}{\partial y} + \beta\frac{\partial \alpha}{\partial y}\right) = \alpha\frac{\partial \beta}{\partial y}.$$

So that

$$(k-m)x^{k-mn-1} = -\beta\frac{\partial \alpha}{\partial y}.$$

Therefore, $k = m$, which contradicts $k > mn$.

Suppose, now, that $m = 1$. Hence, (5.5) becomes

$$kx^{k-n-1}\beta + \beta\frac{\partial f_{k-n}}{\partial y} = (f_{k-n} + x^{k-n-1}y)\frac{\partial \beta}{\partial y}. \tag{5.8}$$

Hence, $f_{k-n} = -x^{k-n-1}y + \alpha\beta$. Taking this into (5.8), and cancelling all common terms, we get

$$(k-1)x^{k-n-1} = -\beta\frac{\partial \alpha}{\partial y}.$$

But this implies that $k = 1 < n$, a contradiction.

Second case: $k = mn$.

By Lemma 5.1, $f_k = \beta^m$. Equating homogeneous components of degree $k-1$, we have that

$$m\beta^{m-1}\frac{\partial \beta}{\partial x} + \beta\frac{\partial f_{k-n}}{\partial y} = mf_{k-n}\frac{\partial \beta}{\partial y}. \tag{5.9}$$

Once again, there are two subcases. Assume first that $m > 1$. Then β divides f_{k-n}, so that $f_{k-n} = \beta^s h$, with $\gcd(\beta, h) = 1$. Taking this into (5.9) we find that

$$m\beta^{m-1}\frac{\partial \beta}{\partial x} + \beta\left(s\beta^{s-1}h\frac{\partial \beta}{\partial y} + \beta^s\frac{\partial h}{\partial y}\right) = m\beta^s h\frac{\partial \beta}{\partial y}.$$

Equating degrees on both sides of this equation

$$n(m-1) = sn + \deg(h), \tag{5.10}$$

so that $s \leq m-1$. However, if $s < m-1$ then, cancelling β^s throughout the equation, we get

$$m\beta^{m-1-s}\frac{\partial \beta}{\partial x} + sh\frac{\partial \beta}{\partial y} + \beta\frac{\partial h}{\partial y} = mh\frac{\partial \beta}{\partial y}. \tag{5.11}$$

This implies that β divides $(m-s)h\partial\beta/\partial y$, which can only happen if $m = s$. The contradiction shows that $s = m-1$ which, together with (5.10), implies that $\deg(h) = 0$. Taking $h = c \in \mathbb{Q}[i]$ and $s = m-1$ into (5.11), we obtain

$$m\frac{\partial \beta}{\partial x} = c\frac{\partial \beta}{\partial y} \tag{5.12}$$

which contradicts the irreducibility of β, by Lemma 5.3.

Finally, if $m = 1$ then by (5.9)

$$\frac{\partial \beta}{\partial x} + \beta\frac{\partial f_{k-n}}{\partial y} = f_{k-n}\frac{\partial \beta}{\partial y}.$$

Since all the terms of this equation are homogeneous of the same degree, and $\partial\beta/\partial x \neq 0$ has degree $n-1$, it follows that f_{k-n} must have degree zero. Thus, once again, we end up with an equation like (5.12), which contradicts Lemma 5.3 and finishes the proof. □

6. Type 3

In this section we construct an explicit example of a derivation induced by a row of type 3 in $\mathfrak{U}_2$. The proof of the proposition follows the approach introduced by D. Jordan in [11].

Proposition 6.1. *The ring $\mathbb{C}[x,y]$ is d-simple with respect to the derivation*

$$d = (xy+1)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Proof. We proceed by contradiction. Suppose that d has a stable nonconstant irreducible polynomial $f \in \mathbb{C}[x,y]$, and write

$$f = a_m(x)y^m + \cdots + a_1(x)y + a_0(x),$$

where $a_m, \dots, a_0 \in \mathbb{C}[x]$. Thus, $d(f) = gf$, for some $g \in \mathbb{C}[x,y]$ of total degree at most 1. If $m = 0$ then

$$(xy+1)\frac{\partial f}{\partial x} = gf.$$

But $xy+1$ is irreducible in $\mathbb{C}[x,y]$, so it must divide the right hand side. Since $f \in \mathbb{C}[x] \setminus \mathbb{C}$ and g has total degree at most 1, we get a contradiction. Therefore, $m \geq 1$. Thus,

$$d\left(\frac{f}{a_m}\right) = \frac{f}{a_m}\left(g - \frac{d(a_m)}{a_m}\right). \tag{6.1}$$

In other words, $\widehat{f} = f/a_m \in \mathbb{C}(x)[y]$ is stable under d. Let

$$\widehat{f} = y^m + b_{m-1}y^{m-1} + \cdots + b_1 y + b_0,$$

where $b_j = a_j/a_m \in \mathbb{C}(x)$. Denoting differentiation with respect to x by a dash, the term of largest possible degree of $d(\widehat{f})$ as a polynomial in y is

$$xb'_{m-1}y^m,$$

which has degree m in y. Since $\deg_y(\widehat{f}) = m$, it follows from (6.1), that $g - d(a_m)/a_m$ must have degree zero as a polynomial in y. In particular, $g \in \mathbb{C}[x]$.

However, the term of largest possible degree of $d(f)$, as a polynomial in y, is

$$xa'_m y^{m+1},$$

whilst gf has degree m in y. Thus, $a'_m = 0$. Therefore, we may assume, without loss of generality, that $a_m = 1$.

Equating the coefficients of the terms of degree j in y on both sides of $d(f) = gf$, we obtain

$$xa'_{j-1} + a'_j + (j+1)xa_{j+1} = ga_j.$$

When $j = m$ this equation gives $g = xa'_{m-1}$; so it can be rewritten as

$$xa'_{j-1} + a'_j + (j+1)xa_{j+1} = xa'_{m-1}a_j. \tag{6.2}$$

Consider now the case $j = m-1$. We conclude from (6.2) that

$$xa'_{m-2} + a'_{m-1} + mx = xa'_{m-1}a_{m-1},$$

and there are two cases that we must consider. Assume first that $a_{m-1} \in \mathbb{C}$. In this case

$$xa'_{m-2} + mx = 0;$$

so that $a'_{m-2} = -m$. Taking this into (6.2) with $j = m - 2$, we find that

$$xa'_{m-3} - m + (m-1)a_{m-1}x = 0,$$

which implies that $m = 0$, a contradiction. Thus, we can assume that $\deg(a_{m-1}) \geq 1$. In particular, $a'_{m-1} \neq 0$ and

$$\deg(xa'_{m-1}a_j) \geq \deg(a_j) + 1.$$

Hence, if $\deg(a_{j+1}) < \deg(a_j)$, then (6.2) implies that

$$\deg(a'_{j-1}) = \deg(a'_{m-1}) + \deg(a_j).$$

Therefore,

$$a_{j-1} \notin \mathbb{C} \quad \text{and} \quad \deg(a_{j-1}) > \deg(a_j). \tag{6.3}$$

Since

$$\deg(a_{m-1}) \geq 1 > \deg(a_m) = 0,$$

it follows from (6.3) and induction that $a_{-1} \neq 0$, which is a contradiction. □

7. Conclusion

We need a technical lemma before we proceed to the proof of Theorem 1.1.

Lemma 7.1. *The set of polynomials $ax^2+bx+c \in \mathbb{Q}[i][x]$, that are irreducible over $\mathbb{Q}[i][x]$ is dense in the set of all quadratic polynomials with complex coefficients.*

Proof. Let f be a quadratic polynomial in $\mathbb{C}[x]$ and $\epsilon > 0$ a real number. The ∞-norm can be defined in the space of quadratic polynomials by identifying it with $\mathbb{C}^3$ in the usual way. Since a real number can always be approximated by rational numbers, there exists $g = ax^2 + bx + c \in \mathbb{Q}[i][x]$ such that

$$\|f - g\|_\infty = \eta < \epsilon.$$

If g is irreducible, we are done. Thus, we may assume that g is not irreducible over $\mathbb{Q}[i][x]$. But this implies that the discriminant of g is a perfect square; say

$$b^2 - 4ac = \left(\frac{\alpha}{\beta}\right)^2, \tag{7.1}$$

where $\alpha \in \mathbb{Z}[i]$ and $\beta \in \mathbb{Z}$ are co-prime as Gaussian integers.

Now choose a prime $p \in \mathbb{Z}$ such that

- p does not divide β,
- $p \equiv 3 \pmod 4$, and
- $(4ap)^{-1} < \epsilon - \eta$.

Then, the polynomial

$$h_0 = ax^2 + bx + \left(c - \frac{1}{4ap}\right),$$

satisfies

$$\|g - h_0\|_\infty = \frac{1}{4ap} < \epsilon - \eta,$$

so that

$$\|f - h_0\|_\infty \leq \|f - g\|_\infty + \|g - h_0\|_\infty \leq \eta + \epsilon - \eta = \epsilon.$$

However, by (7.1), h_0 has discriminant

$$\Delta = b^2 - 4a\left(c - \frac{1}{4ap}\right) = \left(\frac{\alpha}{\beta}\right)^2 + \frac{1}{p},$$

which is equal to

$$\frac{\alpha^2 p + \beta^2}{p\beta^2}.$$

Moreover,

$$\gcd(\alpha^2 p + \beta^2, \beta) = \gcd(\alpha^2 p, \beta) = 1,$$

since $\gcd(\alpha, \beta) = 1$ and p does not divide β. Therefore, Δ is a perfect square if and only if $p\beta^2$ is a perfect square. But this contradicts the fact that p is a Gaussian prime, which follows from $p \equiv 3 \pmod 4$ and [1, Theorem 5.1(c), p. 406]. □

The next two corollaries are simple consequences of the lemma.

Corollary 7.2. *The space of polynomials $x^2 + bx + c \in \mathbb{Q}[i][x]$, that are irreducible over $\mathbb{Q}[i][x]$ is dense in the set of all quadratic* monic *polynomials with complex coefficients.*

Corollary 7.3. *The set of polynomials $ax^2 + bxy + cy^2 \in \mathbb{Q}[i][x, y]$, that are irreducible over $\mathbb{Q}[i][x, y]$ is dense in the set of all quadratic polynomials with complex coefficients of $\mathbb{C}[x, y]$.*

The proof of Theorem 1.1 follows easily from the results that have already been proved, as we now show.

Proof of Theorem 1.1 (1) has been proved in Proposition 3.4; while (2) is a consequence of Propositions 4.1, 5.4 and 6.1, when $j = 1$, 2, and 3, respectively. For $j = 4$, (2) follows from the fact, proved in [14, Theorem 6.2], that the derivation

$$(x^2 + py)\frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

corresponding to the row $(x^2 + py, 1) \in \mathcal{P}_4$, is simple in $\mathbb{C}[x, y]$ when $p \neq 0$. Thus, we need only prove (3).

By Proposition 3.5 we have only to show that, for $j = 1, 2$, there exists a subset $\mathcal{A}_j$ such that

(i) a generic element of $\mathcal{A}_j$ does not have any algebraic solution; and
(ii) $\dim(\mathrm{Im}(\Psi_j)) = 8$;

where $\Psi_j : G \times \mathcal{A}_j \to \mathcal{P}_j$ is the map defined by $\Psi_j(\gamma, u) = u \circ \gamma$; see section 3. We will define $\mathcal{A}_j$, and prove (i) for rows of types 1 and 2. Then, we will explain how (ii) is shown to be correct for both types of rows using computer algebra methods.

A canonical row of type 1 is of the form

$$(xy + a_1, a_2(xy + a_1) + a_3x^2), \text{ for } a_1, a_2, a_3 \in \mathbb{C},$$

so we can identify $\mathcal{C}_1$ with the set of triples $(a_1, a_2, a_3) \in \mathbb{C}^3$. But, if U is an open subset of $\mathcal{C}_1 \cong \mathbb{C}^3$ then, by Corollary 7.2, there exists a triple

$$(a_1, a_2, a_3) \in \mathbb{Q}[i]^3 \cap U \text{ with } a_1a_2a_3 \neq 0,$$

and such that $y^2 + a_2xy + a_3x^2$ is irreducible over $\mathbb{Q}[i]$. However, by Proposition 4.1, the type 1 derivation corresponding to this unimodular row is simple. Therefore, simple derivations are dense in $\mathcal{C}_1$, which proves (i) for $j = 1$ with $\mathcal{A}_1 = \mathcal{C}_1$.

We now turn to rows of type 2. Let

$$\mathcal{A}_2 = \{(\beta, 1) : \beta \in \mathbb{C}[x, y] \text{ is a homogeneous polynomial of degree 2 }\}.$$

The set of rows $(\beta, 1) \in \mathcal{A}_2$ for which $\beta \in \mathbb{Q}[i][x, y]$ is irreducible, is dense in $\mathcal{A}_2$ by Corollary 7.3. Therefore, by Proposition 5.4,

$$\Delta_2 \cap \mathcal{A}_2 \text{ is dense in } \mathcal{A}_2;$$

which proves (i) for rows of type 2.

Finally, we must turn to the proof of (ii). In order to compute the dimension of $\mathrm{Im}(\Psi_j)$, it is enough to find a Gröbner basis for the ideal of the closure of $\mathrm{Im}(\Psi_j)$. In order to do that, consider Ψ_j as a polynomial parameterization of its image. The implicitization of Ψ_j can then be computed using Gröbner bases methods; see [3, Chapter 3, Section 3, p. 126] for example. For $j = 2$, this takes little more than a second if one uses SINGULAR (version 3.0.1 under Windows XP) running on a microcomputer with an Intel Pentium 4 processor of 2.8 GHz, with 512 MB of primary memory.

Unfortunately, the same computation stalls when $j = 1$. So, in this case, we use a strategy similar to the one already applied in Proposition 3.4; namely, we compute the stabilizer

$$\mathrm{Stb}_G(u, \mathcal{C}_1) = \{\gamma \in G : u \circ \gamma \in \mathcal{C}_1\} \subseteq G$$

with respect to a generic row $u \in \mathcal{C}_1$; that is, a row with undetermined coefficients. The computation is now very fast and returns

$$\dim(\mathrm{Stb}_G(u, \mathcal{C}_1)) = 1.$$

Combining this with [15, Theorem 7, p. 60], we find that

$$1 = \dim(\mathrm{Stb}_G(u, \mathcal{C}_1)) \geq \dim(G \times \mathcal{C}_1) - \dim(\mathrm{Im}(\Psi_1)) = 1;$$

so that

$$\dim(\mathrm{Im}(\Psi_1)) = \dim(G \times \mathcal{C}_1) - 1 = 8,$$

as we wished to prove.

Note that this is the only point in this paper at which computer algebra methods are used as part of a proof. A file with the programs can be downloaded from `http://www.dcc.ufrj.br/~collier`.

References

[1] M. Artin, *Algebra*, Prentice-Hall, (1991).

[2] M. N. Carnicer, *The Poincaré problem in the nondicritical case*, Ann. Math., **140**(1994), 289–294.

[3] D. Cox, J. Little and D. O'Shea, *Ideals, varieties and algorithms: an introduction to computational algebraic geometry and commutative algebra*, first edition, Springer (1992).

[4] S. C. Coutinho, *d-simple rings and simple $\mathcal{D}$-modules*, Math. Proc. Cambridge Phil. Soc. **125**, (1999), 405–415.

[5] S. C. Coutinho and L. Menasché Schechter, *Algebraic solutions of holomorphic foliations: an algorithmic approach*, Journal of Symbolic Computation **41**(5) (2006), 603–618.

[6] T. Daly, *Axiom: the thirty year horizon, volume 1: tutorial*, Lulu Press, 2005.

[7] A. M. Doering, Y. Lequain and C. C. Ripoll, *Differential simplicity and cyclic maximal left ideals of the Weyl algebra $A_2(K)$*, Glasgow Math. J., **48** (2006), 269-274.

[8] K. R. Goodearl and R. B. Warfield, *An introduction to noncommutative noetherian rings*, London Mathematical Society Texts 16, Cambridge University Press (1989).

[9] J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, (1981).

[10] G.-M. Greuel, G. Pfister, and H. Schönemann. SINGULAR 3.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern (2005). `http://www.singular.uni-kl.de`.

[11] D. Jordan, *Differentiably simple rings with no invertible derivatives*, Quart. J. Math. Oxford (2), **32** (1981), 417–424.
[12] Y.-K. Man and M. A. H. MacCallum, *A rational approach to the Prelle-Singer algorithm*, J. Symb. Computation **24** (1997), 31–43.
[13] D. Mumford, *The red book of varieties and schemes*, second expanded edition, Springer (1999).
[14] A. Maciejewski, J. Moulin Ollagnier and A. Nowicki, *Simple quadratic derivations in two variables*, Communications in Algebra **29** (2001) 5095 - 5113.
[15] I. R. Shafarevich, *Basic algebraic geometry*, Springer, Berlin-Heidelberg (1977)
[16] S. Walcher, *On the Poincaré problem*, J. Diff. Equations, **166** (2000), 51–78.
[17] S. Zakeri, *Dynamics of singular holomorphic foliations on the complex projective plane*, Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998), 179–233, Contemp. Math., 269, Amer. Math. Soc., Providence, RI (2001).

Departamento de Ciência da Computação, Instituto de Matemática, Universidade Federal do Rio de Janeiro, P.O. Box 68530, 21945-970 Rio de Janeiro, RJ, Brazil.

Programa de Engenharia de Sistemas e Computação, COPPE, UFRJ, PO Box 68511, 21941-972, Rio de Janeiro, RJ, Brazil.

E-mail address: `collier@impa.br`