

ON HOLOMORPHIC FOLIATIONS WITHOUT ALGEBRAIC SOLUTIONS

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ABSTRACT. We give a symbolic algorithm to check whether certain holomorphic foliations of the complex projective plane have an algebraic solution. This algorithm is then used to produce new examples of foliations without algebraic solutions.

1. INTRODUCTION

The study of algebraic solutions of foliations on the complex projective plane goes back to G. Darboux's paper [Darboux 1878]. One of its main contributors in the nineteenth century was H. Poincaré; at the beginning of [Poincaré 1891], he says:

To find out if an equation of the first order and of the first degree is algebraically integrable, it is enough to find an upper bound for the degree of the integral; after this one need only perform some purely algebraic calculations.

A more recent milestone in the subject was J. P. Jouanolou's lecture notes [Jouanolou 1979]. An often quoted result from these notes states that the set of holomorphic foliations of the complex projective plane \mathbb{P}^2 that do *not* have an algebraic solution is dense in the space that parametrizes the foliations; see [Jouanolou 1979, ch. 4, p. 157ff]. In order to prove this result, Jouanolou had to construct a particular example of a foliation of \mathbb{P}^2 without algebraic solutions. However, both Jouanolou's and later proofs of this fact depend on the many special properties possessed by this example; see [Jouanolou 1979, p. 160ff], [Lins Neto 1988, p. 219ff] and [Cerveau and Lins Neto 1991, p. 90]. Moreover, all the other explicit examples of holomorphic foliations of \mathbb{P}^2 without algebraic solutions that have been constructed are essentially variations on Jouanolou's. See section 4 for more details.

The purpose of this paper is to use the methods of computer algebra to search for new examples of foliations without algebraic solutions. The strategy consists in using the Gröbner bases method to automate Poincaré's 'purely algebraic calculations'. In this way we can write down an example of a foliation that may be expected to have no solutions and check whether that is so using the algorithm.

Of course, Poincaré's strategy depends on one being able to find an upper bound for the degree of the algebraic solution of a given foliation. Although such a bound does not always exist, a sharp bound is known for foliations with no positive rational exponents—see section 2 for the definition. So we must first check whether a foliation is of this type, before we try to find an algebraic solution. Luckily the

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foliations that satisfy this condition form a dense set in the space that parametrizes holomorphic foliations of \mathbb{P}^2 .

In section 2 we describe the strategy behind the algorithm, and use results from the theory of holomorphic foliations to explain why it works. The part of the algorithm that checks that the foliation does not have positive rational exponents is discussed in section 3. Finally, in section 4 we give new examples of foliations of \mathbb{P}^2 without algebraic solutions. Unlike most other examples these are quite different from Jouanolou's foliation. To check that the examples satisfy the required properties we use an implementation of our algorithms in the computer algebra system SINGULAR; see [Greuel et al. 1998].

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2. HOLOMORPHIC FOLIATIONS

Let $[z_0 : z_1 : z_2]$ be homogeneous coordinates in the complex projective plane \mathbb{P}^2 . A *holomorphic foliation* \mathcal{F} in \mathbb{P}^2 is given by a 1-form $\omega = \sum_{i=0}^2 A_i dz_i$ where A_0, A_1, A_2 are homogeneous polynomials in z_0, z_1, z_2 of degree $k + 1$ such that $\sum_{i=0}^2 z_i A_i$ is identically zero. The *singular set* of \mathcal{F} is the algebraic variety with equations $A_i = 0$ for $0 \leq i \leq 2$. It will be denoted by $\text{Sing}(\mathcal{F})$.

Let U_i be the open set of \mathbb{P}^2 defined by $z_i \neq 0$. Since $U_0 \cong \mathbb{C}^2$, a vector field on \mathbb{C}^2 gives rise to a 1-form on U_0 , and thus to another way of defining a foliation of \mathbb{P}^2 . For let $x_1 = z_1/z_0$ and $x_2 = z_2/z_0$ be coordinates in U_0 and let P_1 and P_2 be polynomials in $\mathbb{C}[x_1, x_2]$ with no common factors. To the vector field $P_1\partial/\partial x_1 + P_2\partial/\partial x_2$ there corresponds the 1-form $\alpha = P_2 dx_1 - P_1 dx_2$ in the open set U_0 . Denote by $\pi : \mathbb{C}^3 \setminus \{z_0 = 0\} \rightarrow U_0$ the map

$$\pi(z_0, z_1, z_2) = (z_1/z_0, z_2/z_0) = (x_1, x_2).$$

The pull-back $\pi^*\alpha$ is a form in \mathbb{C}^3 with poles in $z_0 = 0$. Now choose d so that $\omega = z_0^d \pi^*\alpha$ is a 1-form that is not divisible by z_0 . It is easily checked that ω satisfies all the requirements of the first paragraph, so it determines a foliation of \mathbb{P}^2 . Moreover, it is not difficult to show that all foliations of \mathbb{P}^2 can be defined in this way.

Of course there is nothing special about U_0 , and the same construction holds equally well for both U_1 and U_2 . However, for the sake of consistency we will assume that the foliations in this paper are always defined by vector fields on U_0 .

Conventions. Let \mathcal{F} be a foliation of \mathbb{P}^2 . Throughout the paper the notation $\mathcal{F} = \mathcal{F}(P_1, P_2)$ means that:

- (1) P_1, P_2 are polynomials of $\mathbb{C}[x_1, x_2]$ with no common factors, and
- (2) \mathcal{F} is defined in U_0 by the vector field $P_1\partial/\partial x_1 + P_2\partial/\partial x_2$.

We must define what it means for an algebraic curve S of \mathbb{P}^2 to be invariant under \mathcal{F} . Without loss of generality we may assume that $S \cap U_0 \neq \emptyset$, and that it is the set of zeroes in U_0 of a polynomial $f \in \mathbb{C}[x_1, x_2]$. Let $\mathcal{F} = \mathcal{F}(P_1, P_2)$. Then S is *invariant under \mathcal{F}* when there exists a polynomial $g \in \mathbb{C}[x_1, x_2]$ such that

$$(2.1) \quad P_1\partial f/\partial x_1 + P_2\partial f/\partial x_2 = gf.$$

In this case we also say that S is an *algebraic solution* of \mathcal{F} .

Now let $z \in \mathbb{P}^2$ be a singular point of \mathcal{F} . As in the previous paragraph, we can assume that $z \in U_0$ and that $\mathcal{F} = \mathcal{F}(P_1, P_2)$. Then $P_1(z) = P_2(z) = 0$. Denote by $J_{\mathcal{F}}(z)$ the jacobian of (P_1, P_2) in z . If the eigenvalues of $J_{\mathcal{F}}(z)$ are non-zero then \mathcal{F} is said to be *non-degenerate* at z , and we can compute their ratios. Following Poincaré's terminology in [Poincaré 1891] we call these ratios the *exponents* of the singularity. A *non-degenerate foliation* is one which is non-degenerate at all of its singular points. The set of all exponents of a non-degenerate foliation \mathcal{F} will be denoted by $\text{Exp}(\mathcal{F})$.

We retain, for the moment, the notation of the previous paragraph. Let $k(\mathcal{F}) = \max\{\deg P_1, \deg P_2\}$, and denote by $P_i^{(j)}$ the homogeneous components of degree j of the polynomial P_i . If $k = k(\mathcal{F})$, let

$$\Delta_i(\mathcal{F}) = x_2 P_1^{(k-i)} - x_1 P_2^{(k-i)}.$$

These polynomials play a key rôle in the next theorem, which is at the core of the algorithms we propose. We will denote the line $z_0 = 0$ of \mathbb{P}^2 by L_∞ .

Main Theorem. *Let \mathcal{F} be a non-degenerate foliation of \mathbb{P}^2 .*

- (1) *The line L_∞ is invariant under \mathcal{F} if and only if $\Delta_0(\mathcal{F})$ is a nonzero polynomial.*
- (2) *If $\Delta_0(\mathcal{F})$ is identically zero and $\text{Exp}(\mathcal{F}) \cap \mathbb{Q}^+ = \emptyset$ then any irreducible algebraic curve of \mathbb{P}^2 invariant under \mathcal{F} must have degree less than or equal to $k(\mathcal{F})$.*

The proof of this theorem follows by collating a number of known results in the theory of holomorphic foliations. We give a proof here both for the sake of completeness, and because some of the formulae will be required for our algorithm. For a proof with a more algebraic-geometric bent, see [Bernstein and Lunts 1988, Appendix, theorem 4, p. 241].

In the proof of the theorem we use the *degree* of a foliation; its definition and properties can be found in [Lins Neto 1988, section 1.4].

Proof. Let $\mathcal{F} = \mathcal{F}(P_1, P_2)$. In order to simplify the notation we will write k for $k(\mathcal{F})$ and Δ_i for $\Delta_i(\mathcal{F})$ throughout the proof.

Let $y_1 = z_0/z_1$ and $y_2 = z_2/z_1$ in U_1 . In these coordinates the line L_∞ has equation $y_1 = 0$. Moreover, in $U_0 \cap U_1$ we have that $x_1 = 1/y_1$ and $x_2 = y_2/y_1$. Performing this change of variables in the 1-form $P_1 dx_2 - P_2 dx_1$ and multiplying the resulting form by y_1^{k+2} to cancel the pole, we obtain

$$(2.2) \quad y_1 \hat{P}_1 dy_2 - (y_2 \hat{P}_1 - \hat{P}_2) dy_1,$$

where $\hat{P}_i(y_1, y_2) = y_1^k P_i(1/y_1, y_2/y_1)$. The coefficient of degree j of \hat{P}_i considered as a polynomial in y_1 with coefficients in $\mathbb{C}[y_2]$ is $P_i^{(k-j)}(1, y_2)$. Thus

$$y_2 \hat{P}_1 - \hat{P}_2 = \Delta_0(1, y_2) + y_1 \Delta_1(1, y_2) + y_1^2 S,$$

for some polynomial $S \in \mathbb{C}[x_1, x_2]$.

Suppose first that Δ_0 is a nonzero polynomial. Then $\Delta_0(1, y_2)$ is also nonzero, because Δ_0 is homogeneous in x_1 and x_2 . In particular, y_1 does not divide $y_2 \hat{P}_1 - \hat{P}_2$. Thus \mathcal{F} is defined in U_1 by the vector field $y_1 \hat{P}_1 \partial/\partial y_1 + (y_2 \hat{P}_1 - \hat{P}_2) \partial/\partial y_2$, so that $y_1 = 0$ is invariant under \mathcal{F} .

On the other hand, if $\Delta_0 = 0$ then we can cancel y_1 from (2.2) so that \mathcal{F} is represented in U_1 by the vector field

$$(2.3) \quad \hat{P}_1 \partial / \partial y_1 + (\Delta_1(1, y_2) + y_1 S) \partial / \partial y_2.$$

Now the coefficient of \hat{P}_1 of degree zero in y_1 is $P_1^{(k)}(1, y_2)$. But $y_2 P_1^{(k)} = P_2^{(k)} \neq 0$, so that \hat{P}_1 is not divisible by y_1 . Hence $y_1 = 0$ is not a solution of \mathcal{F} which completes the proof of (1).

We now turn to (2). If $\Delta_0(\mathcal{F}) = 0$ then \mathcal{F} is a foliation of degree $k - 1$ by [Lins Neto 1988, lemma 2, p. 203]. Let S be an irreducible algebraic curve of \mathbb{P}^2 invariant under \mathcal{F} . Since $\text{Exp}(\mathcal{F}) \cap \mathbb{Q}^+ = \emptyset$, it follows from [Soares 1993, lemma 5.1, p. 156] that if S is not smooth then all its singularities are normal crossings. Finally, by [Cerveau and Lins Neto 1991, theorem 1, p. 891] we have that the degree of such a curve S must be smaller than or equal to the degree of the foliation plus 1. Hence $\deg(S) \leq k$ and the proof is complete.

Corollary. *We retain the notation and hypotheses of the main theorem. Suppose that $\Delta_0(\mathcal{F}) = 0$. Then:*

- (1) *both P_1 and P_2 have degree exactly $k(\mathcal{F})$;*
- (2) *the singularities of \mathcal{F} in $L_\infty \cap U_1$ are given by*

$$P_1^{(k)}(1, y_2) = \Delta_1(1, y_2) = 0;$$

- (3) *The trace and the determinant of the jacobian of \mathcal{F} at a point of $(0, y_2) \in U_1 \cap \text{Sing}(\mathcal{F})$ are given by $t_1(1, y_2)$ and $d_1(1, y_2)$, respectively, where*

$$\begin{aligned} t_1(x_1, x_2) &= \Delta_2 + \partial P_1^{(k-1)} / \partial x_2 \\ d_1(x_1, x_2) &= \Delta_2 \partial P_1^{(k)} / \partial x_2 - P_1^{(k-1)} \partial \Delta_1 / \partial x_2. \end{aligned}$$

- (4) *$[0 : 0 : 1]$ is a singular point of \mathcal{F} if and only if $P_2^{(k)}(0, 1) = \Delta_1(0, 1) = 0$. Moreover, if $[0 : 0 : 1] \in \text{Sing}(\mathcal{F})$ then the trace and the determinant of the jacobian at this point are given by $t_2(0, 1)$ and $d_2(0, 1)$, respectively, where*

$$\begin{aligned} t_2(x_1, x_2) &= \partial \Delta_1 / \partial x_1 - P_2^{(k-1)} \\ d_2(x_1, x_2) &= \Delta_2 \partial P_2^{(k)} / \partial x_1 - P_2^{(k-1)} \partial \Delta_1 / \partial x_1. \end{aligned}$$

Proof. Let $k(\mathcal{F}) = \max\{\deg P_1, \deg P_2\}$. Clearly if $\Delta_0 = x_2 P_1^{(k)} - x_1 P_2^{(k)}$ is identically zero then both P_1 and P_2 must have degree k , thus proving (1). Since $\Delta_0 = 0$, it follows from equation (2.3) that a singular point of \mathcal{F} in $L_\infty \cap U_1$ must satisfy the equations

$$P_1^k(1, y_2) = \Delta_1(1, y_2) = 0,$$

which proves (2). The formulae in (3) follow from the computation of the partial derivatives of \hat{P}_1 and $\Delta_1(1, y_2) + y_1 S$ at $(0, y_2)$. Note that except possibly for $[0 : 0 : 1]$, all the singular points of \mathcal{F} at L_∞ belong to U_1 . Finally, (4) follows from an argument analogous to the above, applied to $[0 : 0 : 1] \in U_2$.

The main theorem suggests the following strategy for checking whether a given foliation has invariant algebraic curves:

First stage: Check whether \mathcal{F} is a non-degenerate foliation with no positive rational exponents. If it is not, then we cannot be certain that the third stage will work properly.

Second stage: Check whether $\Delta_0(\mathcal{F})$ is zero or nonzero. If nonzero then L_∞ is invariant under \mathcal{F} , if zero then proceed to the third stage.

Third stage: Let $\mathcal{F} = \mathcal{F}(P_1, P_2)$. Check whether there is a non-constant polynomial f of degree d , where $0 < d \leq k(\mathcal{F})$, and a polynomial g of degree $k(\mathcal{F}) - 1$ such that equation (2.1) holds. If no such polynomials exist then \mathcal{F} has no invariant curves.

The second stage is completely straightforward, and the third stage is also easy to implement. We use the naïve approach which is already found in [Darboux 1878, section 2, p. 71ff]. It consists in choosing generic polynomials for f and g , and equating coefficients on both sides of (2.1). Gröbner bases can then be used to check whether the system of equations admits a solution.

For the next algorithm it will be necessary to choose a term order in the set of monomials in x_1 and x_2 for which the constants have degree zero. We will make use of the usual multi-index notation, so that if $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ then $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$.

Solution Algorithm.

Input: polynomials P_1 and P_2 of $\mathbb{C}[x_1, x_2]$.

Output: the degree of an algebraic solution of $\mathcal{F}(P_1, P_2)$, or 0 if \mathcal{F} does not have a solution of degree less than or equal to $k(\mathcal{F})$.

- (1) Let $k = \max\{\deg P_1, \deg P_2\}$ and compute $\Delta_0 = x_2 P_1^{(k)} - x_1 P_2^{(k)}$. If $\Delta_0 \neq 0$ return 1.
- (2) Let $f = \sum_{|\alpha| \leq k} u_\alpha x^\alpha$ and $g = \sum_{|\beta| \leq k-1} v_\beta x^\beta$ and construct the ideal I generated by the polynomials in the u s and v s that are the coefficients of the monomials x^α in $P_1 \partial f / \partial x_1 + P_2 \partial f / \partial x_2 - g f$.
- (3) Let $\mathcal{U} = \{x^\alpha : |\alpha| \leq k\}$.
- (4) Choose the element x^α of highest order in \mathcal{U} , and let $J_{\mathcal{U}}$ be the ideal generated by I and $u_\alpha - 1$.
- (5) If $1 \notin J_{\mathcal{U}}$ then return $|\alpha|$ and stop.
- (6) Add u_α to the generators of I , remove x^α from \mathcal{U} and go back to step 4.

Two important points about the algorithm should be observed. First, we can make $u_\alpha = 1$ in step 4 because equation (2.1) is homogeneous in the u s. Second, since f represents a constant polynomial when $\mathcal{U} = \{1\}$, it follows that $J_{\mathcal{U}} \neq (1)$ in this case. In particular the algorithm always stops.

Of course the algorithm merely searches for an algebraic solution with degree less than or equal to the degree of the foliation. However, if the foliation has positive rational exponents then these may not exhaust all the possible solutions.

Since we are using Gröbner bases only to check whether a given ideal is trivial or not, we can choose any term order we want for this algorithm. In the next section we show how Gröbner bases can be used to handle the first stage of the algorithm sketched above.

Finally, it has been shown in [Carnicer 1994] that an upper bound on the degree of an algebraic solution in terms of the degree of the foliation also exists for the more general case of a *nondicritical foliation*. It would be very interesting to find a way of using Gröbner bases to check nondicriticalness. For the definition see [Carnicer 1994] and [Cano 1988]—one should be warned that this is not quite the same as the ‘classical’ nondicritical condition.

3. RATIONAL EXPONENTS

The purpose of this section is to show how Gröbner bases can be used to implement the first stage of the algorithm sketched in section 2. We will sum up the various steps by which this is achieved in the algorithm at the end of the section.

Let $x_1 = z_1/z_0$ and $x_2 = z_2/z_0$ be coordinates in the open set U_0 of \mathbb{P}^2 , and let $\mathcal{F} = \mathcal{F}(P_1, P_2)$ as in the conventions of section 2. Let $k = k(\mathcal{F}) = \max\{\deg P_1, \deg P_2\}$ and $\Delta_0 = \Delta_0(\mathcal{F}) = x_2 P_1^{(k)} - x_1 P_2^{(k)}$. As in section 2, we will denote by $J_{\mathcal{F}}$ the jacobian matrix of (P_1, P_2) . Thus $\text{tr} J_{\mathcal{F}}$ and $\det J_{\mathcal{F}}$ are both polynomials in $\mathbb{C}[x_1, x_2]$. We will consider separately the singularities in U_0 , and those in L_{∞} .

Suppose first that $u \in U_0$ is a singularity of \mathcal{F} . The worst case occurs when u is a degenerate singular point; that is, when one of the eigenvalues of $J_{\mathcal{F}}(u)$ is zero. To find out if there are any degenerate singular points it is enough to compute a Gröbner basis for the ideal generated by P_1, P_2 and $\det J_{\mathcal{F}}$. If the basis is not equal to $\{1\}$, then \mathcal{F} has degenerate singularities, and some of its exponents will not even be well-defined.

So we may assume, from now on, that \mathcal{F} has no degenerate singular points in U_0 . Let λ_1 and λ_2 be the eigenvalues of $J_{\mathcal{F}}(u)$ at some singular point $u \in U_0$ of \mathcal{F} . Since we are assuming that both eigenvalues are nonzero, it follows that the exponent $\eta = \lambda_1/\lambda_2$ is a well-defined complex number. Now

$$\frac{\text{tr}(J_{\mathcal{F}}(u))^2}{\det J_{\mathcal{F}}(u)} = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1 \lambda_2} = \eta + \frac{1}{\eta} + 2.$$

Thus if

$$\theta_0(x_1, x_2, s) = (\det J_{\mathcal{F}})s^2 + (2 \det J_{\mathcal{F}} - (\text{tr} J_{\mathcal{F}})^2)s + \det J_{\mathcal{F}},$$

and $u = (u_1, u_2)$, then $\theta_0(u_1, u_2, \eta) = 0$.

The calculations above suggest that we should compute a Gröbner basis G of the ideal generated by the polynomials P_1, P_2 and θ_0 with respect to the lexicographical order that has $s < x_1 < x_2$. Since the polynomials P_1 and P_2 have no common factor, the system $P_1 = P_2 = 0$ has only finitely many solutions. Hence the same must hold for $P_1 = P_2 = \theta_0 = 0$. By [Adams and Loustanau 1994, corollary 2.2.11, p. 65] there must be a polynomial $e_0(s)$ in G that contains only the variable s . In particular, if a complex number is an exponent of a singular point of \mathcal{F} in U_0 then it is a root of $e_0(s)$. Thus if $e_0(s)$ has no positive rational roots, then \mathcal{F} has no positive rational exponents at those of its singularities that belong to U_0 .

This settles the problem so far as singularities in U_0 are concerned. But, of course, \mathcal{F} could have singularities in L_{∞} . One way to deal with these is to repeat all the above for the vector fields that represent \mathcal{F} in U_1 and in U_2 . However, since the computations of a Gröbner bases under the lexicographical order can be rather slow it is convenient to use the formulae in the corollary of section 2 to speed up the calculations.

First note that, by the corollary, \mathcal{F} has singularities in $L_{\infty} \cap U_1$ if and only if the polynomials $P_1^{(k)}(1, y_2)$ and $\Delta_1(1, y_2)$ have a common root. Since these are polynomials in one variable, we can check whether they are co-prime using the Euclidean algorithm.

Assuming that the polynomials have a common root, we must find whether \mathcal{F} has degenerate singularities on $L_{\infty} \cap U_1$ and, if not, whether there are any positive rational exponents. The general strategy is the same as the one used for singularities

in U_0 , and more details can be found in the description of the algorithm given below. Note that the rôle of θ_0 is played, in this case, by the polynomial

$$\theta_1(y_2, s) = d_1(1, y_2)s^2 + (2d_1(1, y_2) - t_1(1, y_2)^2)s + d_1(1, y_2),$$

where $t_1(x_1, x_2)$ and $d_1(x_1, x_2)$ are defined in (3) of the corollary of section 2. Finally, we must determine whether $[0 : 0 : 1]$ is a singularity of \mathcal{F} , and compute its exponent if it is non-degenerate. To do this we use (4) of the corollary of section 2 just as we used (2) and (3) for the singularities in $L_\infty \cap U_1$.

Next we translate these ideas into an algorithm which determines whether a given foliation has any positive rational exponents.

Rational Exponents Algorithm.

Input: polynomials P_1 and P_2 of $\mathbb{C}[x_1, x_2]$.

Output: *true* if $\mathcal{F}(P_1, P_2)$ has no positive rational exponents and *false* otherwise.

- (1) Use Gröbner bases to check whether the ideal generated by P_1 , P_2 and $\det J_{\mathcal{F}}$ contains 1. If it does not return *false* and stop.
- (2) Define $\theta_0(x_1, x_2, s)$ as above and use Gröbner bases with respect to the lexicographical order $s < x_1 < x_2$ to find a polynomial $e_0(s)$ in the variable s alone, that belongs to the ideal generated by P_1 , P_2 and θ_0 .
- (3) Check whether $e_0(s)$ has a positive rational root, if it does return *false*.
- (4) Compute $g(y_2) = \gcd(P_1^{(k)}(1, y_2), \Delta_1(1, y_2))$.
- (5) If $g = 1$ return *true*; otherwise compute $\gcd(g(y_2), d_1(1, y_2))$ —see the corollary of section 2 for the definition of d_1 . If it is not 1, return *false*.
- (6) Define $\theta_1(y_2, s)$ as above, and use Gröbner bases, with respect to the lexicographical order $s < y_2$, to compute a polynomial $e_1(s)$ on the variable s alone, that belongs to the ideal generated by

$$P_1^{(k)}(1, y_2), \quad \Delta_1(1, y_2) \quad \text{and} \quad \theta_1(y_2, s).$$

- (7) Check whether $e_1(s)$ has a positive rational root, if it does return *false*.
- (8) Compute $P_2^{(k)}(0, 1)$ and $\Delta_1(0, 1)$. If one of them is nonzero, return *true*; if both are zero and $d_2(0, 1) = 0$, return *false*.
- (9) Compute

$$\theta_2(s) = (d_2(0, 1)s^2 + (2d_2(0, 1) - t_2(0, 1)^2)s + d_2(0, 1)).$$

If $\theta_2(s)$ has a positive rational root, return *false*; otherwise, return *true*.

When implementing this algorithm it is a good idea to check first whether the system $P_1 = P_2 = 0$ has dimension zero or not. Of course if it does not, then the polynomials do not determine a foliation in the sense of section 2.

Our implementation of this algorithm using the computer algebra system SINGULAR has two peculiarities that should be mentioned. First, to check whether the one variable polynomials $e_0(s)$ and $e_1(s)$ have a positive rational root we first factorize them using the SINGULAR command `factorize`. Second, when computing Gröbner basis with respect to the lexicographical order we use the SINGULAR command `stdfglm` to speed up the calculations; see [Greuel et al. 1998, p. 236] and [Adams and Loustanaou 1994, pp. 67 and 68, exercises 2.2.7 and 2.2.8] for the algorithm behind this command.

4. THE EXAMPLES

Although it is known that the set of foliations of \mathbb{P}^2 without algebraic solutions is dense, a search of the literature reveals a dearth of concrete examples. In fact, all examples are variations of the Jouanolou foliation; see [Jouanolou 1979, p. 160ff]. A very interesting family of foliations of this sort is the one discovered by Żołądek in [Żołądek 1998]. Unlike the foliations considered in this paper, the foliations in Żołądek's family can have degenerate singularities for some choice of parameters.

However, both Jouanolou's foliation and the non-degenerate foliations in the examples of Żołądek are of *Poincaré type*; that is, none of their exponents are positive real numbers. In fact, in all these examples the exponents of the non-degenerate foliations are not even real numbers. For the relevance of the Poincaré hypothesis to this problem, see [Lins Neto 1988] or [Cerveau and Lins Neto 1991].

In this section we give an example of a simple foliation that we have shown to have no algebraic solutions using our implementation of the algorithms *solution* and *rational exponents* in the computer algebra system SINGULAR. It turns out that, unlike the previously known examples, this foliation has exponents that are positive real numbers; so they are not of Poincaré type.

Let

$$P_1 = 2x_1^3 + 3x_1^2x_2 + ax_2^2 + 1 \quad \text{and} \quad P_2 = 2x_1^2x_2 + 3x_1x_2^2 + bx_1^2 + 1,$$

where $a \cdot b \neq 0$ and $8a + 27b \neq 0$. Using Gröbner bases and the method of undetermined coefficients one shows that $x_1P_2 - x_2P_1$ is an irreducible polynomial of degree 3 if $a \neq b$. In particular $\gcd(P_1, P_2) = 1$, and we get a family of foliations $\mathcal{F}(P_1, P_2)$. Since this family is parametrized by a and b we will denote it by $\mathcal{F}(a, b)$ for short. One readily checks that $\Delta_0(\mathcal{F}(a, b)) = 0$ and that $\Delta_1(\mathcal{F}(a, b)) = ax_2^3 - bx_1^3$. In particular, the line at infinity is not invariant under \mathcal{F} . Moreover, the system

$$P_1^{(k)}(1, y_2) = 2 + 3y_2 = 0 \quad \text{and} \quad \Delta_1(1, y_2) = ay_2^3 - b = 0,$$

does not have a solution because $8a + 27b \neq 0$. Since we also have that $\Delta_1(0, 1) = a \neq 0$, we conclude that $\text{Sing}(\mathcal{F}) \subset U_0$.

Let $\mathbb{I} = \{(a, b) \in \mathbb{Z}^2 : a \neq b \text{ and } 1 \leq a, b \leq 10\}$. Using a computer and the *rational exponents* algorithm it takes less than a minute to check that for $(a, b) \in \mathbb{I}$ the foliation $\mathcal{F}(a, b)$ has a rational exponent only when $a = 9$ and $b = 4$. However, even in this case the exponent is -1 , a *negative* rational number. On the other hand, using a combination of Gröbner bases and numerical computations in MAPLE V.5, we have shown that if $(a, b) \in \mathbb{I}$ and $\mathcal{F}(a, b)$ has a positive real exponent then either $b = 1$, or $a = 3$ and $b = 2$. Moreover all these real exponents occur at a real singularity. Finally, using an implementation of the *solutions* algorithm in SINGULAR it is easy to check that $\mathcal{F}(a, b)$ does not have an algebraic solution for any $(a, b) \in \mathbb{I}$.

Let us illustrate the output of the algorithms when $a = 2$ and $b = 3$. In the notation used to describe the *rational exponents* algorithm, we have that

$$\begin{aligned} e_0(s) = & 757707876887s^{14} - 1515415753774s^{13} - 1582513020691s^{12} \\ & - 6026827836972s^{11} + 11777436443567s^{10} + 29426173550670s^9 + \\ & 5040018620749s^8 - 81748114201576s^7 + 5040018620749s^6 + \\ & 29426173550670s^5 + 11777436443567s^4 - 6026827836972s^3 - \\ & 1582513020691s^2 - 1515415753774s + 757707876887. \end{aligned}$$

This polynomial is irreducible over \mathbb{Q} , so it has no rational roots. Using the *solution* algorithm we find that there is no invariant curve of degree less than or equal to 3. Hence by the theorem of section 2 we conclude that $\mathcal{F}(2, 3)$ has no algebraic solutions.

Now, it turns out that $e_0(s)$ has two positive real roots, namely $0.7386877578\dots$ and its reciprocal. Although all exponents of $\mathcal{F}(2, 3)$ are roots of $e_0(s)$, the converse need not be true. However, a numerical computation shows that one of the singularities of $\mathcal{F}(2, 3)$ has real coordinates, and these roots turn out to be the two exponents of this singularity. In view of these results, it seems reasonable to pose the following two questions:

- Are there any non-zero complex numbers a and b , for which $(a - b)(8a + 27b) \neq 0$ and $\mathcal{F}(a, b)$ has an algebraic solution?
- Is the set of all (a, b) for which $\mathcal{F}(a, b)$ has a real exponent dense in \mathbb{C}^2 ?

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