ON THE DIFFERENTIAL SIMPLICITY OF POLYNOMIAL RINGS

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ABSTRACT. Commutative differentially simple rings have proved to be quite useful as a source of examples in non-commutative algebra. In this paper we use the theory of holomorphic foliations to construct new families of derivations with respect to which the polynomial ring over a field of characteristic zero is differentially simple.

1. INTRODUCTION

In noncommutative algebra, simple noetherian rings have posed many difficult problems. In the absence of two-sided ideals most of the tools used by ring theorists break down. This has meant that teasing apart the structure of simple rings can be a very difficult task. For this reason, new families of examples of simple rings are always welcome as new testing grounds for old conjectures.

Most examples of simple rings are constructed as iterated Ore extensions. For a given commutative ring R and a derivation d of R, the Ore extension $R[x; d]$ is is defined by the equation $rx - xr = d(r)$, for every $r \in R$. However, if R has a proper nonzero ideal I such that $d(I) \subseteq I$, then $IR[x; d]$ is a proper nonzero ideal of $R[x; d]$. Such an ideal I is said to be *invariant* under d. Hence, for $R[x; d]$ to be simple, R must not have any proper nonzero ideals invariant under d. In this case we say that R is a *d-simple ring*. It turns out that this condition is also sufficient, so that $R[x; d]$ is simple if and only if R is d-simple; see [7, Propostion 1.14, p. 15] for a proof.

One might reasonably ask whether all commutative rings are d-simple for some derivation d. The answer is no. Indeed, it is not difficult to see that if R is d-simple, then Ω^1_R must have R as a direct summand. Thus, if a polynomial ring $R = K[x_1, \dots, x_n]$, over a field K, is d-simple with respect to $d = \sum_{i=1}^n g_i \partial/\partial x_i$ then $(g_1, \dots, g_n)$ is a unimodular row. In other words, there exist polynomials $h_1, \dots, h_n$ such that $\sum_{i=1}^n h_i g_i = 1$.

Rather few examples of d-simple rings are known. This is partly due to the fact that it can be difficult to check whether a given derivation has proper nonzero invariant ideals. One of the first examples of a d-simple ring to come to light was the polynomial ring in two variables over a field of characteristic zero. G. Bergman showed that this ring is d-simple with respect to $d = \partial/\partial x + (1 + xy)\partial/\partial y$. Since then a few more examples of derivations with respect to which polynomial rings are

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d-simple have been constructed, but they are far from plentiful; see [10] and [1], for example.

In this paper we propose a method that can be used to construct new examples of derivations with respect to which $\mathbb{C}[x, y]$ is d-simple. The method is based on a theorem of Carnicer [3] that establishes an upper bound on the degree of the generator of an invariant principal ideal. However, for this bound to hold, the foliation of the complex projective plane $\mathbb{P}^2$ induced by the derivation d of $\mathbb{C}[x, y]$ must be nondicritical. The method is described in more detail at the end of section 2, which begins with a summary of the results of the theory of holomorphic foliations that will be needed in the paper. These include the definition of nondicritical foliations and a brief discussion of Carnicer's theorem and its consequences.

The challenge then consists in finding unimodular rows over $\mathbb{C}[x, y]$ which give rise to nondicritical foliations of $\mathbb{P}^2$. This turns out to be an easier task then one would expect. A ready source of examples of unimodular rows over a ring R is the elementary group $E_n(R)$. This is the subgroup of $G\ell_n(R)$ generated by the matrices of the form $1 + re_{ij}$, where $r \in R$ and e_{ij} is the $n \times n$ matrix with 1 in the ij position and zeroes elsewhere. In sections 3 and 4 we explore the 2×2 elementary group as a source of derivations which give rise to nondicritical foliations

However, as has been shown by P. M. Cohn in [4, proposition 7.3, p. 26], not all unimodular rows over the polynomial ring in two variables come from the elementary group. In section 5 we give an example of a unimodular row that cannot be a row of an elementary matrix, but which gives rise to a derivation with respect to which $\mathbb{C}[x, y]$ is d-simple.

The final section highlights some of the difficulties one encounters in trying to generalise these results.

2. Non-dicritical foliations

In this section we review some basic facts about holomorphic foliations and prove a number of results on the special type of foliation that will appear later on in the paper. For a very nice introduction to the theory of holomorphic foliations over the complex projective plane see [15].

Let S be a smooth irreducible complex algebraic surface, and let $\mathcal{F}$ be a singular holomorphic foliation of S by curves. Given a point $p \in S$, a holomorphic neighbourhood U_p of p is said to be *adapted* to $\mathcal{F}$ if:

(1) U_p has coordinates x and y;
(2) $p = (0, 0)$ in these coordinates;
(3) $\mathcal{F}$ is defined on U_p by a vector field

$$a(x, y)\partial/\partial x + b(x, y)\partial/\partial y \tag{2.1}$$

where $a(x, y)$ and $b(x, y)$ are holomorphic functions on U.

It follows from the definition of a singular holomorphic foliation that every point of S admits an adapted neighbourhood. Denote by $a_i(x, y)$ and $b_i(x, y)$ the i-th components of the Taylor expansions of a and b at p. The *multiplicity* of $\mathcal{F}$ at p is the positive integer

$$\mu_p(\mathcal{F}) = \min\{i : a_i \text{ or } b_i \text{ is nonzero}\}.$$

If $\mu_p(\mathcal{F}) \geq 1$, then p is a *singularity* of $\mathcal{F}$. The set of singularities of $\mathcal{F}$ will de denoted by $\mathrm{Sing}(\mathcal{F})$. We will assume throughout the paper that this set is finite.

Given $p \in \mathrm{Sing}(\mathcal{F})$, denote by $\mathcal{F}'$ the strict transform of $\mathcal{F}$ under the blowup $\pi : S' \to S$ with centre p. If the exceptional divisor $\pi^{-1}(p)$ is invariant under $\mathcal{F}'$ then the blowup is said to be *nondicritical*; otherwise it is called *dicritical*. It is very easy to check whether a blowup is nondicritical, as the following result shows. For a proof see [13, 489ff].

Proposition 2.1. *Let $p \in \mathrm{Sing}(\mathcal{F})$ and suppose that U_p is a neighbourhood of p adapted to $\mathcal{F}$. Then $\mathcal{F}$ is nondicritical with respect to the blowup $\pi : S' \to S$ with centre p if and only if*

$$xb_\mu(x, y) - ya_\mu(x, y) \neq 0$$

where $\mu = \mu_p(\mathcal{F})$.

Suppose, as before, that $p \in \mathrm{Sing}(\mathcal{F})$ and that U_p is an adapted neighbourhood of p. Let λ_1 and λ_2 be the eigenvalues of the 1-jet of the vector field (2.1). Then p is a *simple* singularity of $\mathcal{F}$ if $\lambda_2 \neq 0$ and either

(1) $\lambda_1 = 0$, or
(2) λ_1/λ_2 is not a positive rational number.

Given a singular holomorphic foliation on a surface, we can always resolve its singularities into simple singularities by a succession of blowups. More precisely, if $p \in S$ is a singular point of $\mathcal{F}$ there is a resolution

$$S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S,$$

such that

(1) π_j is a blowup of S_{i-1} with centre p_{i-1};
(2) $p_0 = p$;
(3) $\mathcal{F}_0 = \mathcal{F}$;
(4) if $\mathcal{F}_i$ is the strict transform of $\mathcal{F}_{i-1}$ under π_i, then p_i is a singular point of $\mathcal{F}_i$ for $i = 1, \dots, n-1$; and,
(5) if $q \in S_n$ projects to p under $\pi_1\pi_2\cdots\pi_n$ then, either q is a simple singularity of $\mathcal{F}_n$, or $\mathcal{F}_n$ is nonsingular at q.

If each one of the blowups in this resolution is nondicritical, then p is a *nondicritical* singular point of $\mathcal{F}$. Moreover, this condition is independent of the resolution. Nondicritical singularities are easy to come by, as the next proposition shows; for a proof see [13, (2.b), p.517]

Proposition 2.2. *Let p be a singular point of the foliation $\mathcal{F}$ of S. If*

(1) p is a simple singularity, or
(2) the 1-jet of the vector field that defines $\mathcal{F}$ at p has the form $(x+y)\partial/\partial x + y\partial/\partial y$,

then p is a nondicritical singular point of $\mathcal{F}$.

We now turn to foliations of the complex projective plane. Let $a(x, y)$ and $b(x, y)$ be polynomials of degrees n and m, respectively. The derivation $a(x, y)\partial/\partial x + b(x, y)\partial/\partial y$ of $\mathbb{C}[x, y]$ gives rise to the 1-form $bdx - ady$ of $\mathbb{C}^2$. Denote by U_z the open subset of $\mathbb{P}^2$ defined by $z \neq 0$. The open sets U_x and U_y are defined similarly. Let $\pi : U_z \to \mathbb{C}^2$ be the map defined by $\pi([x : y : z]) = (x/z, y/z)$. The pull-back $\pi^*(bdx - ady)$ is a meromorphic 1-form with poles in $z = 0$. Choose a positive integer r such that

$$\omega_{a,b} = z^r \pi^*(bdx - ady) \tag{2.2}$$

is a 1-form with no poles in $\mathbb{P}^2$, that is not divisible by z. This form defines a foliation of $\mathbb{P}^2$ that we denote by $\mathcal{F}(a,b)$. Conversely, every holomorphic foliation of $\mathbb{P}^2$ can be defined in this way.

Let C be a reduced curve in $\mathbb{P}^2$ defined as the zero set of the homogeneous polynomial $F(x,y,z) \in \mathbb{C}[x,y,z]$. We say that C is *invariant* under $\mathcal{F}(a,b)$ if there exists a homogeneous 2-form θ such that

$$\omega_{a,b} \wedge dF = F\theta.$$

This is equivalent to saying that there exists a polynomial $g \in \mathbb{C}[x,y]$ such that

$$a(x,y)\partial f/\partial x + b(x,y)\partial f/\partial y = gf$$

where $f(x,y) = F(x,y,1)$ is the dehomogeneization of F at U_z. Such a reduced curve C is also called an *algebraic solution* of $\mathcal{F}(a,b)$.

It turns out that there is a very simple criterion for checking whether the line L_∞ with equation $z=0$ is invariant under $\mathcal{F}(a,b)$. First we need some notation. Let h be a polynomial in $\mathbb{C}[x,y]$, we will denote by h_i the ith homogeneous component of h. For every positive integer i, write

$$\Delta_i(\mathcal{F}(a,b)) = ya_i - xb_i.$$

For a proof of the next proposition see [11, Lemma 2(b), p. 203]

Proposition 2.3. *Let a and b be polynomials in $\mathbb{C}[x,y]$ and denote by k the number $\max\{\deg(a), \deg(b)\}$. The line L_∞ is invariant under $\mathcal{F}(a,b)$ if and only if the polynomial $\Delta_k(\mathcal{F}(a,b))$ is nonzero.*

We may now define the degree of the foliation $\mathcal{F} = \mathcal{F}(a,b)$ as follows:

$$\deg(\mathcal{F}) = \begin{cases} k & \text{if } \Delta_k(\mathcal{F}) \neq 0; \\ k-1 & \text{if } \Delta_k(\mathcal{F}) = 0. \end{cases}$$

The key result to most of our proofs is the following theorem of M. M. Carnicer in [3].

Theorem 2.4. *Let $\mathcal{F}$ be a foliation of the complex projective plane $\mathbb{P}^2$. If C is a reduced algebraic curve invariant under $\mathcal{F}$ and there are no dicritical singularities of $\mathcal{F}$ in C then*

$$\deg(C) \leq \deg(\mathcal{F}) + 2.$$

The result we need is a corollary of Carnicer's theorem. A foliation is called *nondicritical* if it has only nondicritical singularities.

Corollary 2.5. *Let $\mathcal{F}$ be a nondicritical foliation of $\mathbb{P}^2$ which leaves L_∞ invariant. If C is a reduced algebraic curve invariant under $\mathcal{F}$ and L_∞ is not a component of C then*

$$\deg(C) \leq \deg(\mathcal{F}) + 1.$$

Moreover, $\mathcal{F}$ has only finitely many reduced algebraic solutions.

Proof. Suppose that C is the zero set of the reduced homogeneous polynomial $F \in \mathbb{C}[x,y,z]$. Since L_∞ is not a component of C, it follows that zF is invariant under $\mathcal{F}$ and reduced. But

$$\deg(zF) = \deg(F) + 1 \leq \deg(\mathcal{F}) + 2$$

by theorem 2.4. Hence

$$\deg(C) = \deg(F) \leq \deg(\mathcal{F}) + 1,$$

as desired. The fact that there are only finitely many solutions for $\mathcal{F}$ follows immediately from the existence of an upper bound for the degree of a reduced curve invariant under $\mathcal{F}$.

We will now turn to the special kind of foliation that we use in this paper. Throughout the rest of the section we assume that

$$n = \deg(a) \geq m = \deg(b) \quad \text{and that} \quad \Delta_n(\mathcal{F}(a,b)) \neq 0.$$

Since $ya_n - xb_n \neq 0$, the pole of $\pi^*(bdx - ady)$ has order $n+2$. A simple computation shows that the 1-form $z^{n+2}\pi^*(bdx - ady)$ is equal to

$$\omega_{a,b} = z^{n-m+1}b^h(x,y,z)dx - za^h(x,y,z)dy + (ya^h(x,y,z) - xz^{n-m}b^h(x,y,z))dz,$$

where $a^h(x,y,z)$ denotes the homogeneization of $a(x,y)$. Note that the line at infinity L_∞ is invariant under $\mathcal{F}(a,b)$.

Now suppose that (a,b) is a unimodular row of $\mathbb{C}[x,y]^2$. Then the singularities of $\mathcal{F}(a,b)$ are given by the equations

$$z = 0 \text{ and } ya^h(x,y,z) - xz^{n-m}b^h(x,y,z) = 0.$$

Moreover, $a^h(x,y,0) = a_n(x,y)$ the homogeneous component of maximal degree of a, and $b^h(x,y,0) = b_m(x,y)$.

There are two cases that we must consider. First let $n = m$. In this case the singularities are defined by

$$ya_n(x,y) - xb_n(x,y) = z = 0.$$

The second case corresponds to $n > m$; and the singularities satisfy the equations

$$ya_n(x,y) = z = 0.$$

We now come to the main result of this section.

Theorem 2.6. *Let a and b be polynomials of $\mathbb{C}[x,y]$ of degrees n and $n-1$, respectively. Suppose that*

(1) (a,b) is a unimodular row,
(2) $a_n(1,0)a_n(0,1) \neq 0$,
(3) $b_{n-1}(1,0) \neq 0$, and
(4) $a_n(1,y)$ has no double roots.

Then $\mathcal{F}(a,b)$ is a nondicritical foliation of $\mathbb{P}^2$.

Proof. As we have seen above, the hypotheses of the theorem imply that the singularity set of $\mathcal{F}(a,b)$ is given in homogeneous coordinates by

$$\{[1:0:0]\} \cup \{[1:y:0] : a_n(1,y) = 0\}.$$

Hence all singularities of $\mathcal{F}(a,b)$ belong to the open set U_x. But in this open set the foliation is defined by the vector field

$$(ya^h(1,y,z) - zb^h(1,y,z))\partial/\partial y + za^h(1,y,z)\partial/\partial z.$$

The jacobian matrix of this vector field at a point with coordinates $[1:y:0]$ is

$$\begin{bmatrix} a_n(1,y) + y\partial a_n(1,y)/\partial y & 0 \\ ya_{n-1}(1,y) - b_{n-1}(1,y) & a_n(1,y) \end{bmatrix}$$

Thus at the singular point $[1:0:0]$ the jacobian is

$$\begin{bmatrix} a_n(1,0) & 0 \\ -b_{n-1}(1,0) & a_n(1,0) \end{bmatrix}.$$

Since we are assuming that $b_{n-1}(1,0) \neq 0$, it follows from proposition 2.2 that the foliation is nondicritical at this singular point.

Assuming now that the singularity is $[1 : y_0 : 0]$ and that $a_n(1, y_0) = 0$, the jacobian is

$$\begin{bmatrix} y_0 \partial a_n(1,y_0)/\partial y & 0 \\ y_0 a_{n-1}(1,y_0) - b_{n-1}(1,y_0) & 0 \end{bmatrix}$$

Since $y_0 \neq 0$ and $a_n(1, y)$ has no double roots, it follows that $y_0 \partial a_n(1, y_0)/\partial y \neq 0$. Thus these singularities are simple and nondicritical by proposition 2.2, and the proof is complete.

We may now summarise the method which is used in the next sections. The first step consists in concocting an adequate family $\mathcal{D}$ of derivations $d = a\partial/\partial x + b\partial/\partial y$ of $\mathbb{C}[x, y]$. For the sake of simplicity we assume that $n = \deg(a) > \deg(b)$, but the method also works when $\deg(a) = \deg(b)$, as we shall see in section 5. One must then find a subfamily $\mathcal{D}_0$ of $\mathcal{D}$ such that, for all $d \in \mathcal{D}_0$ the following two conditions are satisfied:

(1) the foliation $\mathcal{F}(a, b)$ of $\mathbb{P}^2$ is nondicritical;
(2) if f is a polynomial of degree k invariant under d then f_k is divisible by a factor of degree $n - 1$ of a_n.

We can use (2) to prove that d has no solutions of degree less than or equal to $n + 1$. It then follows from (1) and corollary 2.5 that d has no algebraic solutions at all. The subfamily $\mathcal{D}_0$ may be defined either by a generic condition (leading to a set that is dense in $\mathcal{D}$) or by an effective condition. The later has the advantage of producing very concrete examples with coefficients over $\mathbb{Q}$.

3. Key lemmas

Throughout this section we will abide by the following notation and hypotheses.

Hypotheses 3.1. *Let $n \geq r \geq 3$ be an integer and let $\lambda = x + y$. Assume that α and β are polynomials in $\mathbb{C}[x, y]$ such that:*

(3.1.1) $\deg \beta = n$, $\beta_{n-i} = 0$ *for* $1 \leq i \leq r + 2$ *and* $\beta_n(1, 0)\beta_n(0, 1) \neq 0$*;*
(3.1.2) $\deg(\alpha) = n - r$*;*
(3.1.3) λ does not divide $\alpha_{n-r}\beta_n$;
(3.1.4) β has no common zeroes with α;
(3.1.5) no linear factor of β_n has multiplicity greater than 1*;*
(3.1.6) y does not divide α_{n-r};
(3.1.7) β_n has no factors in common with α_{n-r}.

Write $\mathcal{F}$ for the foliation $\mathcal{F}(\lambda\beta + \alpha, \beta)$.

Let $X \neq L_\infty$ be an algebraic curve in $\mathbb{P}^2$ that is invariant under $\mathcal{F}$. If X is the set of zeroes of a homogeneous polynomial F of degree $k > 0$, then the affine curve $X \cap U_z$ is the set of zeroes of $f(x, y) = F(x, y, 1)$. Moreover, X is invariant under $\mathcal{F}$ if and only if

$$(\lambda\beta + \alpha)\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} = gf, \tag{3.1}$$

for some polynomial $g \in \mathbb{C}[x,y]$ of degree n.

The proofs in this section rely on the formulae obtained by equating homogeneous components of the degree $n+k-i$ on both sides of (3.1). The shape of these formulae depends on i. Thus, if $i=0$

$$\lambda\beta_n \frac{\partial f_k}{\partial x} = g_n f_k; \tag{3.2}$$

while for $1 \le i \le r$, the corresponding formula is

$$\lambda\beta_n \frac{\partial f_{k-i}}{\partial x} + \beta_n \frac{\partial f_{k-i+1}}{\partial y} = \sum_{j=0}^{i} g_{n-j} f_{k-i+j}. \tag{3.3}$$

Finally, if $i=r+1$ or $i=r+2$, we obtain

$$\lambda\beta_n \frac{\partial f_{k-i}}{\partial x} + \sum_{j=r}^{i-1} \alpha_{n-j} \frac{\partial f_{k-i+1+j}}{\partial x} + \beta_n \frac{\partial f_{k-i+1}}{\partial y} = \sum_{j=0}^{i} g_{n-j} f_{k-i+j}. \tag{3.4}$$

Proposition 3.2. *$\mathcal{F}$ is a nondicritical foliation of $\mathbb{P}^2$ and $\mathrm{Sing}(\mathcal{F}) \subset L_\infty$.*

Proof. In order to prove that the singularities of $\mathcal{F}$ belong to L_∞ it is enough to show that $\lambda\beta+\alpha$ and β have no common zeroes. But this follows from hypothesis (3.1.4).

We use theorem 2.6 to prove that $\mathcal{F}$ is nondicritical. In the notation of that theorem we have $a = \lambda\beta+\alpha$ and $b=\beta$. Moreover, $a_{n+1} = \lambda\beta_n$ and $b_n = \beta_n$. Thus

$$a_{n+1}(1,0)a_{n+1}(0,1) = \lambda(1,0)\lambda(0,1)\beta_n(1,0)\beta_n(0,1),$$

and $b_n(1,0) = \beta_n(1,0)$ are nonzero by hypothesis (3.1.1). Since (3.1.3) and (3.1.5) imply that $\lambda\beta_n$ cannot have double roots, it follows from theorem 2.6 that $\mathcal{F}$ is nondicritical.

We begin by collecting some results whose proofs would interrupt the development of later arguments.

Lemma 3.3. *Each linear factor of f_k divides $y\beta_n\lambda$.*

Proof. Let ℓ be a linear factor of f_k. Thus we can write $f_k = \ell^e q$, where $\gcd(\ell, q) = 1$ and e is a positive integer. It follows from (3.2) that

$$\lambda\beta_n \left(eq\frac{\partial \ell}{\partial x} + \ell \frac{\partial q}{\partial x} \right) = g_n \ell q.$$

Hence, if ℓ does not divide $\lambda\beta_n$, then $\partial\ell/\partial x = 0$. But in this case ℓ is a constant multiple of y, which is what we wanted to prove.

Lemma 3.4. *If β_n divides f_k then $\gcd(\beta_n, g_n) = 1$.*

Proof. Suppose that $f_k = \beta_n\phi$, for some polynomial ϕ. It follows from (3.2) that

$$\phi g_n = \lambda \partial f_k/\partial x = \lambda\left(\beta_n \partial\phi/\partial x + \phi\partial\beta_n/\partial x\right). \tag{3.5}$$

Thus, if ℓ is a common linear factor of β_n and g_n, then ℓ divides $\phi\partial\beta_n/\partial x$ by hypothesis (3.1.3). But every linear factor of β_n has multiplicity one by hypothesis (3.1.5), therefore ℓ divides ϕ. Let $\phi = \ell^t\overline{\phi}$, with $\gcd(\overline{\phi}, \ell) = 1$, and $\beta_n = \ell\overline{\beta_n}$. Taking these in (3.5) and cancelling ℓ^t throughout the resulting equation, we obtain

$$g_n\overline{\phi} = \lambda\left(\overline{\beta_n}\ell\partial\overline{\phi}/\partial x + t\overline{\beta_n}\,\overline{\phi}\partial\ell/\partial x + \overline{\phi}\ell\partial\overline{\beta_n}/\partial x + \overline{\beta_n}\,\overline{\phi}\partial\ell/\partial x\right).$$

Since ℓ divides g_n it follows that it also divides

$$(t+1)\overline{\beta}_n\overline{\phi}\partial\ell/\partial x.$$

However, ℓ divides neither $\overline{\beta_n}$ nor $\overline{\phi}$, and $t \geq 0$. Therefore $\partial\ell/\partial x = 0$, which implies that ℓ is a multiple of y. Hence, by hypothesis (3.1.1), ℓ does not divide β_n, a contradiction. It follows that β_n and g_n are co-prime.

Lemma 3.5. *Suppose that $f_k = \beta_n\lambda\theta$. Let $\Xi = \partial\beta_n/\partial x - \partial\beta_n/\partial y$. If θ is a linear form, then:*

(1) $\gcd(\lambda, g_n) = 1$*;*
(2) $\Xi \neq 0$ *and* $\gcd(\beta_n, \Xi) = 1$.

Moreover, (1) holds also if θ is a nonzero constant.

Proof. Taking $\phi = \lambda\theta$ in (3.5) and cancelling λ from both sides of the equation, we obtain

$$\theta g_n = \beta_n\theta + \lambda\partial(\beta_n\theta)/\partial x. \tag{3.6}$$

Suppose now that λ divides g_n and let us aim at a contradiction. It follows from (3.6) that λ divides $\theta\beta_n$, which implies that λ divides θ by hypothesis (3.1.3). Without loss of generality we may assume that $\theta = \lambda$. Taking this into account, and cancelling λ from both sides of (3.6), we get

$$g_n = 2\beta_n + \lambda\partial\beta_n/\partial x. \tag{3.7}$$

Since we are assuming that λ divides g_n, it follows that λ divides β_n, a contradiction. This proves (1). The last statement is similarly proved.

In order to prove (2) we apply Euler's relation to $x\Xi$, which gives

$$x\Xi = n\beta_n - \lambda\partial\beta_n/\partial y. \tag{3.8}$$

Hence, $\Xi \neq 0$ by (3.1.3). Now suppose that ℓ is a common linear factor of Ξ and β_n. It follows from (3.8) and hypothesis (3.1.3) that ℓ divides $\partial\beta_n/\partial y$. Thus, by hypothesis (3.1.5), ℓ cannot be a factor of β_n.

Proposition 3.6. *There is no solution of equation* (3.1) *with* $\gcd(\beta_n, f_k) = 1$.

Proof. Assuming that $\gcd(\beta_n, f_k) = 1$, it follows from (3.3) and induction that β_n divides g_{n-i} for $0 \leq i \leq r$. But $\deg(\beta_n) = n$, while $\deg(g_{n-i}) = n - i$. Hence β_n divides g_n and $g_{n-i} = 0$ for $1 \leq i \leq r$. Thus, we get from (3.4) with $i = r+1$, that

$$\lambda\beta_n\partial f_{k-r-1}/\partial x + \alpha_{n-r}\partial f_k/\partial x + \beta_n\partial f_{k-r}/\partial y = g_n f_{k-r-1} + g_{n-r-1}f_k.$$

Therefore, β_n divides $\alpha_{n-r}\partial f_k/\partial x - g_{n-r-1}f_k$. But by lemma 3.3, $\gcd(\beta_n, f_k) = 1$ implies that $f_k = \lambda^s y^{k-s}$, for some integer $s \geq 0$. Hence

$$\alpha_{n-r}\partial f_k/\partial x - g_{n-r-1}f_k = y^{k-s}\lambda^{s-1}\left(s\alpha_{n-r} - g_{n-r-1}\lambda\right).$$

But this polynomial can only be divisible by β_n if $s\alpha_{n-r} = g_{n-r-1}\lambda$. Since λ does not divide α_{n-r} by hypothesis (3.1.3), it follows that $s = 0$. In particular, $f_k = y^k$.

However, taking $f_k = y^k$ in (3.2), we obtain

$$g_n y^k = \lambda\beta_n\partial(y^k)/\partial x = 0.$$

Hence $g_n = 0$. If we now put

$$i = 1, \quad f_k = y^k, \quad g_n = 0, \quad \text{and} \quad g_{n-1} = 0,$$

then it follows from (3.3) that

$$\lambda\beta_n\partial f_{k-1}/\partial x = -k\beta_n y^{k-1}.$$

But this implies that $k = 0$, which is a contradiction.

Proposition 3.7. *There is no solution of equation* (3.1) *with* $f_k = \beta_n\phi$, *where* $\deg(\phi) \leq 2$.

Proof. Assume that $f_k = \beta_n\phi$ and that $\deg(\phi) \leq 2$. Note that this implies that $k \leq n+2$. It follows, by induction, from Lemma 3.4 and equation (3.3) that

$$\beta_n \text{ divides } f_{k-i} \text{ for } 1 \leq i \leq r. \tag{3.9}$$

Taking $i = r+1$ in (3.4) we conclude that β_n divides

$$\alpha_{n-r}\partial f_k/\partial x - g_n f_{k-r-1}. \tag{3.10}$$

Multiplying (3.10) by λ and taking (3.5) into account, we conclude that β_n divides

$$g_n\left(\alpha_{n-r}\phi - \lambda f_{k-r-1}\right).$$

Thus, by lemma 3.4, β_n also divides $\alpha_{n-r}\phi - \lambda f_{k-r-1}$. But $\alpha_{n-r}\phi - \lambda f_{k-r-1}$ has degree less than or equal to $n-1$, therefore it can only be divisible by β_n if it is zero. Hence, $\alpha_{n-r}\phi = \lambda f_{k-r-1}$, and λ must divide ϕ by hypothesis (3.1.3).

Therefore, we may assume from now on that $f_k = \beta_n\lambda\theta$ for some nonzero homogeneous polynomial θ of degree less than or equal to 1. Taking $\phi = \lambda\theta$ in equation (3.5) it follows that

$$\theta g_n = \partial f_k/\partial x = \theta\partial(\lambda\beta_n)/\partial x + \lambda\beta_n\partial\theta/\partial x. \tag{3.11}$$

Hence, either θ divides $\lambda\beta_n$ or θ is a constant multiple of y.

Taking $i = 1$ and $f_k = \beta_n\lambda\theta$, in (3.3), we get that

$$\lambda\beta_n\partial f_{k-1}/\partial x + \beta_n\left(\lambda\partial\beta_n\theta/\partial y + \beta_n\theta\right) = g_n f_{k-1} + g_{n-1}\lambda\beta_n\theta.$$

But, by (3.9), $f_{k-1} = \beta_n\ell$, where ℓ is a homogeneous polynomial of degree less than or equal to 1. Thus, dividing the equation by β_n, we obtain

$$\lambda\partial f_{k-1}/\partial x + \lambda\partial\beta_n\theta/\partial y + \beta_n\theta = g_n\ell + g_{n-1}\lambda\theta. \tag{3.12}$$

Hence λ divides $\theta\beta_n - g_n\ell$.

If θ is not a constant multiple of λ then $g_n \equiv \beta_n \pmod{\lambda}$ by (3.11), so λ divides $\beta_n(\theta-\ell)$. This implies, by hypothesis (3.1.3), that λ divides $\theta - \ell$. On the other hand, if $\theta \in \mathbb{C}\lambda$, then we may assume without loss of generality that $\theta = \lambda$. Thus $g_n \equiv 2\beta_n \pmod{\lambda}$, and we conclude that λ divides $\theta - 2\ell$. Therefore,

1. if $\theta \notin \mathbb{C}\lambda$ then λ divides $\theta - \ell$;
2. if $\theta = \lambda$ then λ divides $\theta - 2\ell$.

Moreover, $\deg(\theta) = \deg(\ell) \leq 1$, unless $\theta = \lambda$, when we can also have $\ell = 0$.

We will subdivide the remainder of the proof into several cases, each one corresponding to a possible value for θ. Since $\deg(\theta) \leq 1$, it follows from lemma 3.3 that if θ is not a constant then it must divide $y\lambda\beta_n$. Thus, if θ is not a constant, then it is either y, or λ, or a linear factor of β_n.

First case: θ is a nonzero constant.

We may assume, without loss of generality, that $\theta = 1$. Since, in this case, $k = n+1$, then $f_{n+1} = \lambda\beta_n$, and $f_n = \ell\beta_n$. However, by (1), $\ell = \theta = 1$. Hence by (3.9), we have that

$$f_{n+1} = \lambda\beta_n,\ f_n = \beta_n, f_{n-1} = \cdots = f_{n-r+1} = 0 \text{ and } \quad g_n = \beta_n + \lambda\partial\beta_n/\partial x. \tag{3.13}$$

It follows from (3.12) that

$$g_{n-1} = \partial\beta_n/\partial y.$$

We conclude from (3.13) and (3.3) (with $i = 2$) that $g_{n-2} = 0$. A similar argument shows that $g_{n-j} = 0$ for all $2 \leq j \leq r$.

Putting $i = r+1$ in (3.4) and taking into account all the above formulae, we get that

$$\lambda\beta_n\partial f_{n-r}/\partial x + \alpha_{n-r}g_n = g_n f_{n-r} + g_{n-r-1}\lambda\beta_n. \tag{3.14}$$

In particular, $\lambda\beta_n$ divides $g_n(\alpha_{n-r} - f_{n-r})$. Thus, by lemmas 3.4 and 3.5, $\lambda\beta_n$ divides $\alpha_{n-r} - f_{n-r}$. Since $\lambda\beta_n$ has degree $n+1$, this is possible only if $\alpha_{n-r} = f_{n-r}$. Taking $f_{n-r} = \alpha_{n-r}$ in (3.14) and cancelling all common terms, we conclude that $g_{n-r-1} = \partial\alpha_{n-r}/\partial x$.

Therefore, (3.4) with $i = r+2$ gives rise to

$$\lambda\beta_n\partial f_{n-r-1}/\partial x + \alpha_{n-r}\partial\beta_n/\partial x + \alpha_{n-r-1}g_n + \beta_n\partial\alpha_{n-r}/\partial y = \\ g_n f_{n-r-1} + \alpha_{n-r}\partial\beta_n/\partial y + \beta_n\partial\alpha_{n-r}/\partial x + g_{n-r-2}\lambda\beta_n.$$

Write Ξ for $\partial\beta_n/\partial x - \partial\beta_n/\partial y$. Since $g_n = \beta_n + \lambda\partial\beta_n/\partial x$, we conclude that β_n divides

$$\alpha_{n-r}\Xi + \lambda\partial\beta_n/\partial x\,(\alpha_{n-r-1} - f_{n-r-1}).$$

But β_n is homogeneous. Applying Euler's relation we find that

$$\lambda\partial\beta_n/\partial x = (x+y)\partial\beta_n/\partial x = n\beta_n + y\Xi.$$

Thus β_n divides

$$\Xi\,(\alpha_{n-r} + y(\alpha_{n-r-1} - f_{n-r-1})).$$

However, by lemma 3.5, $\Xi \neq 0$ and $\gcd(\beta_n, \Xi) = 1$. Hence, β_n must divide $\alpha_{n-r} + y(\alpha_{n-r-1} - f_{n-r-1})$. Since, $\deg(\beta_n) = n$, it follows that

$$\alpha_{n-r} = -y(\alpha_{n-r-1} - f_{n-r-1}).$$

Therefore, y divides α_{n-r}, contradicting hypothesis (3.1.6).

Second case: $\theta = y$.

Note that, in this case, $k = n+2$. Moreover,

$$f_{n+2} = y\lambda\beta_n,\ f_{n+1} = \ell\beta_n \text{ and } \quad g_n = \beta_n + \lambda\partial\beta_n/\partial x,$$

where $\deg\ell = 1$. Taking these formulae in (3.3) with $i = 1$, and cancelling β_n throughout the resulting equation, we get

$$\lambda\,(\beta_n\partial\ell/\partial x + \ell\partial\beta_n/\partial x) + y\partial\beta_n\lambda/\partial y + \beta_n\lambda = \ell\,(\beta_n + \lambda\partial\beta_n/\partial x) + g_{n-1}y\lambda.$$

Since $\lambda = x+y$ and y does not divide β_n by hypothesis (3.1.1), we conclude that

$$x\partial\ell/\partial x + x - \ell = -y\partial\ell/\partial y + x$$

is divisible by y, which is a contradiction.

Third case: $\theta = \lambda$.

In this case, $f_k = \lambda^2\beta_n$ and $f_{k-1} = \ell\beta_n$, for some linear form ℓ, which might be zero. Thus it follows from (3.3) that

$$\lambda\left(\beta_n\partial\ell/\partial x + \ell\partial\beta_n/\partial x\right) + \left(2\lambda\beta_n + \lambda^2\partial\beta_n/\partial y\right) = g_n\ell + g_{n-1}\lambda^2. \tag{3.15}$$

Hence, by lemma 3.5, λ divides ℓ; say $\ell = c\lambda$. Cancelling λ throughout (3.15), we obtain

$$(c+2)\beta_n + c\lambda\partial\beta_n/\partial x + \lambda\partial\beta_n/\partial y = g_n c + g_{n-1}\lambda. \tag{3.16}$$

But, in this case, $g_n = 2\beta_n + \lambda\partial\beta_n/\partial x$ by (3.7). Cancelling common terms from (3.16), we obtain

$$(2-c)\beta_n + \lambda\partial\beta_n/\partial y = g_{n-1}\lambda.$$

Since λ does not divide β_n, this implies that $c = 2$ and $g_{n-1} = \partial\beta_n/\partial y$. Hence, $f_{k-1} = 2\lambda\beta_n$. Thus, taking $i = 2$ and $f_n = d\beta_n$, for some constant d, in (3.3) and dividing through by β_n, we find that

$$\begin{aligned} &d\lambda\beta_n\partial\beta_n/\partial x + 2\beta_n\left(\beta_n + \lambda\partial\beta_n/\partial y\right) \\ &\qquad = \left(2\beta_n + \lambda\partial\beta_n/\partial x\right)d\beta_n + 2\lambda\beta_n\partial\beta_n/\partial y + g_{n-2}\lambda^2\beta_n. \end{aligned}$$

Cancelling all common terms

$$g_{n-2}\lambda^2 = 2\beta_n(1-d).$$

Since λ does not divide β_n by hypothesis (3.1.3), it follows that $d = 1$. Thus $g_{n-2} = 0$ and $f_n = \beta_n$. But $f_{n-i} = 0$ for $1 \le i \le r-2$, by (3.9), and a similar argument shows that $g_{n-j} = 0$ for all $2 \le j \le r$.

Now, from (3.4) and (3.7), we have that

$$\lambda\beta_n\partial f_{(n+2)-r-1}/\partial x + \lambda\alpha_{n-r}g_n = g_n f_{(n+2)-r-1} + g_{n-r-1}\lambda^2\beta_n. \tag{3.17}$$

Hence, by lemma 3.4, β_n must divide $f_{n-r+1} - \alpha_{n-r}\lambda$. However, $\deg(\beta_n) = n$, so that $f_{(n+2)-r-1} = \alpha_{n-r}\lambda$. Replacing $f_{(n+2)-r-1}$ by $\alpha_{n-r}\lambda$ in (3.17) and cancelling all common terms we get that

$$\alpha_{n-r} + \lambda\partial\alpha_{n-r}/\partial x = \lambda g_{n-r-1},$$

which implies that λ divides α_{n-r}. But this contradicts hypothesis (3.1.3).

Fourth case: θ divides β_n.

Writing $f_{n+1} = \ell\beta_n$ as above we have from (3.3) with $i = 1$ that

$$\lambda\left(\ell\partial\beta_n/\partial x + \beta_n\partial\ell/\partial x\right) + \theta\partial\lambda\beta_n/\partial y + \lambda\beta_n\partial\theta/\partial y = g_n\ell + g_{n-1}\theta\lambda.$$

Thus θ divides

$$\lambda\ell\partial\beta_n/\partial x - g_n\ell.$$

But by (3.2) $g_n = \beta_n + \lambda(\overline{\beta_n}\partial\theta/\partial x + \partial\beta_n/\partial x)$, where $\overline{\beta_n}$ is the co-factor of θ in β_n. Thus θ must divide $\ell\lambda\overline{\beta_n}\partial\theta/\partial x$. This implies that either $\partial\theta/\partial x = 0$ or ℓ is a constant multiple of θ. In the first case θ is a constant multiple of y, which contradicts hypothesis (3.1.1).

In the second case it follows from (1) and hypothesis (3.1.3) that $\ell = \theta$. Taking this in (3.3) with $i = 1$, and eliminating all common terms we find that

$$g_{n-1} = \partial\beta_n/\partial y + \overline{\beta_n}\partial\theta/\partial y = 2\overline{\beta_n}\partial\theta/\partial y + \theta\partial\overline{\beta_n}/\partial y.$$

Write $f_n = d\beta_n$ as before. Putting $i = 2$ in (3.3) and taking into account the previously deduced formulae, we obtain

$$d(\beta_n + \lambda\overline{\beta_n}\partial\theta/\partial x) + g_{n-2}\lambda = 0.$$

Together with hypothesis (3.1.3) this implies that $d = g_{n-2} = 0$. A similar argument shows that $g_{n-j} = 0$ for $2 \leq j \leq r$. Taking all these in (3.4) we deduce that

$$\lambda\beta_n \frac{\partial f_{(n+2)-r-1}}{\partial x} + \alpha_{n-r}\left(\theta\beta_n + \lambda\frac{\partial\theta\beta_n}{\partial x}\right) = \left(\beta_n + \lambda\overline{\beta_n}\frac{\partial\theta}{\partial x} + \lambda\frac{\partial\beta_n}{\partial x}\right) f_{(n+2)-r-1} + g_{n-r-1}\lambda\theta\beta_n. \tag{3.18}$$

But this implies that $\overline{\beta_n}$ divides

$$\lambda\theta\partial\overline{\beta_n}/\partial x(\theta\alpha_{n-r} - f_{(n+2)-r-1}).$$

Thus, by hypotheses (3.1.3) and (3.1.5) $\overline{\beta_n}$ divides $\theta\alpha_{n-r} - f_{(n+2)-r-1}$. But $\overline{\beta_n}$ has degree $n-1$ whilst $\theta\alpha_{n-r} - f_{(n+2)-r-1}$ has degree $n - r + 1 \leq n - 2$. Hence $f_{(n+2)-r-1} = \theta\alpha_{n-r}$. Thus, after elimination of common terms, (3.18) becomes

$$\alpha_{n-r}\partial\theta/\partial x + \theta\partial\alpha_{n-r}/\partial x = g_{n-r-1}\theta.$$

Since θ does not divide α_{n-r} by (3.1.7), it follows that θ is a multiple of y. But this contradicts (3.1.1).

4. The Main theorems

Throughout this section we will assume that $\lambda = x + y$ and that $n \geq 4$ is an integer. We denote by V_n the homogeneous component of degree n of $\mathbb{C}[x,y]$ and by S_n the homogeneous component of degree n of $\mathbb{C}[x,y,z]$. These are naturally isomorphic to the vector spaces $\mathbb{C}^{n+1}$ and $\mathbb{C}^{\binom{n+2}{n}}$, respectively. From now on we identify homogeneous polynomials with their corresponding vectors under these isomorphisms.

To every pair $(\beta, b) \in V_n \times \mathbb{C}$ we associate the 1-form $\omega_{\lambda\beta+b,\beta}$ as in (2.2). Let W_n be the set of homogeneous 2-forms of degree n and let $Y_{n,k}$ be the subset of $V_n \times \mathbb{C} \times W_n \times S_k$ defined by

$$Y_{n,k} = \{(\beta, b, \theta, f) : \omega_{\beta,b} \wedge df = f\theta, (\beta, b, \theta) \neq 0 \text{ and } f \neq 0\}.$$

Note that if $(c_1, c_2) \in \mathbb{C}^* \times \mathbb{C}^*$ and $(\beta, b, \theta, f) \in Y_{n,k}$ then

$$(c_1\beta, c_1 b, c_1\theta, c_2 f) \in Y_{n,k}.$$

Let $\omega = \omega_{\lambda\beta+b,\beta}$. Given the algebraic nature of the equation

$$\omega \wedge df = f\theta, \tag{4.1}$$

it follows that $\overline{Y}_{n,k} = Y_{n,k}/(\mathbb{C}^* \times \mathbb{C}^*)$ is a projective subvariety of $\mathbb{P}(V_n \times \mathbb{C} \times W_n) \times \mathbb{P}(S_k)$.

Moreover, if $(\beta, b, \theta, f) \in Y_{n,k}$, then $(\beta, b) \neq 0$ and $\theta \neq 0$. The first inequality follows directly from equation (4.1). In order to prove the second inequality, contract (4.1) with respect to the Euler vector field E. This gives $-kf\omega = f(\theta \llcorner E)$ from which $\theta \neq 0$ follows immediately. Therefore, the projection

$$V_n \times \mathbb{C} \times W_n \times S_k \to V_n \times \mathbb{C}$$

induces, under passage to the quotient, a morphism $\phi : \overline{Y}_{n,k} \to \mathbb{P}(V_n \times \mathbb{C})$ of projective varieties.

Since the line L_∞ is a solution of $\mathcal{F}(\lambda\beta + b, \beta)$, it follows that, for every nonzero complex number c and integer $k \geq 0$, the polynomial cz^k is also a solution of this foliation. Now a homogeneous polynomial of degree k in the variables x, y and z can be written in the form

$$\sum_{r+s+t=k} a_{rst} x^r y^s z^t.$$

Let $\mathcal{C}_k$ be the closed subvariety of $\overline{Y}_{n,k}$ defined by

$$\mathcal{C}_k = \{[\beta : b : \theta] \times f \in \overline{Y}_{n,k} : a_{rst} = 0 \text{ for } (r,s,t) \neq (0,0,k)\}.$$

Since θ is completely determined by β, b and f in (4.1), it follows that

$$\dim(\mathcal{C}_k) = \dim(\mathbb{P}(V_n \times \mathbb{C})) = n+1. \tag{4.2}$$

Finally, let $\mathcal{V}$ be the closed set of $\mathbb{P}(V_n \times \mathbb{C}) \times \mathbb{P}^2$ defined by

$$\mathcal{V} = \{[\beta : b] \times p : \beta(p) = z = 0\}.$$

Lemma 4.1. *The projective variety $\mathcal{V}$ is irreducible and has dimension equal to $n+1$.*

Proof. Denote by $v_0, \dots, v_n, u$ the homogeneous coordinates of $\mathbb{P}(V_n \times \mathbb{C})$. Let I be the ideal of the polynomial ring $R = \mathbb{C}[x, y, z, v_0, \dots, v_n, u]$ generated by z and by

$$\beta = \sum_{i=0}^{n} v_i x^i y^{n-i}. \tag{4.3}$$

Since β is linear as a polynomial in the vs, it follows that it is irreducible as a polynomial of $R/(z)$. Thus the image of I in $R/(z)$ is a prime ideal. Hence, I is prime in R. Thus $\mathcal{V}$ is irreducible.

On the other hand, $\mathbb{P}(V_n \times \mathbb{C}) \times L_\infty$ is an irreducible variety of dimension $n+2$, of which $\mathcal{V}$ is the hypersurface of equation $\beta = 0$. Hence $\dim(\mathcal{V}) = n+1$.

Writing β as in (4.3), we see that the resultant

$$\mathcal{R}(\beta) = \mathrm{Res}_t(\beta(1,t), \frac{\partial \beta}{\partial y}(1,t))$$

is a polynomial in the vs. Moreover, $\beta(1,t)$ has multiple roots if and only if $\mathcal{R}(\beta) = 0$. Now consider the open set

$$U = \{[\beta : b] \in \mathbb{P}(V_n \times \mathbb{C}) : \beta(1,0)\beta(0,1)\beta(1,-1) \cdot \mathcal{R}(\beta) \cdot b \neq 0\}$$

of $\mathbb{P}(V_n \times \mathbb{C})$. Note that if $[\beta : b] \in U$ then $\mathcal{F}(\lambda\beta + b, \beta)$ satisfies hypotheses (3.1.1) to (3.1.6) of section 3.

Theorem 4.2. *Let $k \geq 2$ be an integer and let $X \neq \mathcal{C}_k$ be an irreducible component of $\overline{Y}_{n,k}$. Then $\phi(X)$ is a proper closed subset of $\mathbb{P}(V_n \times \mathbb{C})$.*

Proof. Since ϕ is a morphism of projective varieties it follows that if $\phi(X) \subsetneq \mathbb{P}(V_n \times \mathbb{C})$ then it is a proper closed subset. Suppose, by contradiction, that $\phi(X) = \mathbb{P}(V_n \times \mathbb{C})$.

Let $\mathcal{X}$ be the closed subset of $\overline{Y}_{n,k} \times \mathbb{P}^2$ defined by

$$\mathcal{X} = \{[\beta : b : \theta] \times f \times p \in \overline{Y}_{n,k} \times \mathbb{P}^2 : f(p) = 0\},$$

and let $\psi : \overline{Y}_{n,k} \times \mathbb{P}^2 \to \mathbb{P}(V_n \times \mathbb{C}) \times \mathbb{P}^2$ be the map induced by $\phi \times id$. Since this is a morphism of projective varieties, $\psi(\mathcal{X})$ is a closed subset of $\mathbb{P}(V_n \times \mathbb{C}) \times \mathbb{P}^2$. Moreover, if

$$\pi_1 : \mathbb{P}(V_n \times \mathbb{C}) \times \mathbb{P}^2 \to \mathbb{P}(V_n \times \mathbb{C})$$

is the projection on the first coordinate, then $\pi_1\psi(\mathcal{X}) = \phi(X)$.

Since $\mathcal{F}(\lambda\beta+b,\beta)$ is nondicritical for all $[\beta : b] \in U$ by proposition 3.2, it follows from corollary 2.5 that $k \leq n+2$ and that the fibres of ϕ over points of U are finite. Thus, by [8, lemma 14.8, p. 178], ϕ induces a finite map $\phi^{-1}(U) \to U$. In particular, $\dim(\phi^{-1}(U)) = \dim(U) = n+1$. Hence, $\mathcal{C}_k$ and $\phi^{-1}(U) \cap X$ are both irreducible and of the same dimension. Together with $\mathcal{C}_k \neq X$, this implies that

$$\dim(\mathcal{C}_k \cap X \cap \phi^{-1}(U)) < n+1.$$

Therefore,

$$\dim(\phi(\mathcal{C}_k \cap X \cap \phi^{-1}(U))) < n+1.$$

In particular, $\phi(\mathcal{C}_k \cap X \cap \phi^{-1}(U)) \subsetneqq U$.

Now, let $[\beta : b]$ be a point of U outside $\phi(\mathcal{C}_k \cap X \cap \phi^{-1}(U))$. This implies that $\mathcal{F}(\lambda\beta+b,\beta)$ satisfies the hypotheses of 3.1. Denoting by $\phi_{|X}$ the restriction of ϕ to X, we have that

$$(\phi_{|X})^{-1}([\beta : b]) = \{[\beta : b : \theta_i] \times f_i : 1 \leq i \leq s\},$$

for some $s \geq 0$. Moreover, it follows from the choice of $[\beta : b]$ that $(\phi_{|X})^{-1}[\beta : b] \cap \mathcal{C}_k = \emptyset$. Hence, no f_i can be a constant multiple of z^k. Denoting by π_2 the projection of $\mathbb{P}(V_n \times \mathbb{C}) \times \mathbb{P}^2$ on the second coordinate, we conclude that

$$Z = \pi_2({\pi_1}^{-1}([\beta : b] \cap \psi(\mathcal{X})) \subset \mathbb{P}^2$$

is the zero set of the product $f_1 \cdots f_s$ in $\mathbb{P}^2$, and $L_\infty \subsetneqq Z$.

Since each curve $f_i = 0$ is invariant under $\mathcal{F}(\lambda\beta+b,\beta)$, so is Z. Thus, by lemma 3.3 and proposition 3.6, $Z \cap \mathcal{B} \neq \emptyset$, where $\mathcal{B} = \{p \in L_\infty : \beta(p) = 0\}$. Hence

$$\pi_1({\pi_1}^{-1}(U) \cap \psi(\mathcal{X}) \cap \mathcal{V}) = U$$

Therefore, by [14, §8, theorem 2, p. 48],

$$\dim(\pi_1^{-1}(U) \cap \psi(\mathcal{X}) \cap \mathcal{V}) \geq \dim(U) = \dim(\mathcal{V}).$$

Since $\mathcal{V}$ is irreducible by lemma 4.1, it follows that $\mathcal{V} \subset \psi(\mathcal{X})$. Moreover, $\pi_1\psi(\mathcal{X}) = \mathbb{P}(V_n \times \mathbb{C})$, by hypothesis. Thus, for every choice of $[\beta : b] \in \mathbb{P}(V_n \times \mathbb{C})$ there exists an algebraic solution f of the foliation $\mathcal{F}(\lambda\beta+b,\beta)$ whose leading homogeneous component is divisible by β. But this contradicts proposition 3.7 thus proving the theorem.

Corollary 4.3. *Let $n \geq 3$ be an integer. There exists a Zariski open set $\mathcal{U}_n$ of $\mathbb{P}(V_n \times \mathbb{C})$ such that if $[\beta : b] \in \mathcal{U}_n$ then $\mathbb{C}[x,y]$ is d-simple with respect to*

$$d = (\lambda\beta + b)\partial/\partial x + \beta\partial/\partial y.$$

Proof. First note that if d has a solution of degree $k \geq 1$ in $\mathbb{C}[x,y]$ then $\mathcal{F}(\lambda\beta+b,\beta)$ has a solution of degree k other than a constant multiple of z^k. Thus $\phi^{-1}([\beta : b]) \cap X \neq \emptyset$ for some component $X \neq \mathcal{C}_k$ of $\overline{Y}_{n,k}$. Together with theorem 4.2 this implies that there exists a *proper* closed subset $F_k \subsetneqq \mathbb{P}(V_n \times \mathbb{C})$ such that if $[\beta : b] \notin F_k$ then the only solution of $\mathcal{F}(\lambda\beta+b,\beta)$ of degree k is z^k. Now let U be the open set defined before theorem 4.2. Set

$$\mathcal{U}_n = U \setminus \bigcup_{k=1}^{n+2} F_k.$$

This is an open Zariski set of $\mathbb{P}(V_n \times \mathbb{C})$. Moreover, if $[\beta : b] \in \mathcal{U}_n$ then L_∞ is the only possible solution of $\mathcal{F}(\lambda\beta+b,\beta)$ with degree less than or equal to $n+2$. But,

$[\beta : b] \in U$ so that $\mathcal{F}(\lambda\beta + b, \beta)$ is nondicritical. Thus by corollary 2.5 $\mathcal{F}(\lambda\beta + b, \beta)$ has no other solutions, and the proof is complete.

Although we have shown that most choices of β give rise to derivations without algebraic solutions, we have not actually exhibited a simple particular example of a derivation with this property. We proceed to fill this gap.

Theorem 4.4. *Let $n \geq 3$ be an integer. Let $\lambda = x + y$ and let β be a homogeneous polynomial of degree n with rational coefficients. If β is irreducible over $\mathbb{Q}$, then for all nonzero $b \in \mathbb{Q}$ the polynomial ring $\mathbb{C}[x, y]$ is d-simple with respect to*

$$d = (\lambda\beta + b)\partial/\partial x + \beta\partial/\partial y.$$

Proof. By [12, proposition 3.3, p. 36], if $\mathcal{F}(\lambda\beta + b, \beta)$ has an algebraic solution other than L_∞, then it has one defined by a polynomial with rational coefficients. Thus we may assume, by contradiction, that there exists a non-constant polynomial $f \in \mathbb{Q}[x, y]$ which satisfies equation (3.1).

But the hypotheses of the theorem imply that (3.1.1) to (3.1.6) hold for $\mathcal{F}(\lambda\beta + b, \beta)$. Hence, this is a non-dicritical foliation, and by corollary 2.5 $\deg(f) \leq n + 2$. Since β is irreducible, it follows from lemma 3.3 and proposition 3.6 that β must divide the leading homogeneous component of f. But this contradicts proposition 3.7, and proves the theorem.

5. A non-elementary example

Let $n \geq 2$ be an integer and let $\theta \in \mathbb{Q}[x, y]$ be a homogeneous polynomial of degree $n - 1$. Then $(y^n, y\theta + 1)$ is a unimodular row of $\mathbb{Q}[x, y]$. However, by [4, proposition 7.3, p. 386] this cannot be a row of an elementary 2×2 matrix over $\mathbb{Q}[x, y]$. We show in this section that, under some extra hypotheses, $\mathbb{C}[x, y]$ is d-simple with respect to

$$(y^n)\partial/\partial x + (y\theta + 1)\partial/\partial y.$$

The following notation will be fixed throughout the section:

$$D_0 = y^n\partial/\partial x + y\theta\partial/\partial y \text{ and } D = D_0 + \partial/\partial y.$$

Moreover, we denote by ζ an nth root of unity, and by σ the automorphism of $\mathbb{C}[x, y]$ defined on a polynomial f by $\sigma(f)(x, y) = f(\zeta x, \zeta y)$.

Lemma 5.1. *If f is an algebraic solution of D then so is $\sigma(f)$*

Proof. Let $h(x, y) = \sigma(f)(x, y)$. Note that

$$\frac{\partial h}{\partial x}(x, y) = \zeta\frac{\partial f}{\partial x}(\zeta x, \zeta y),$$

and that a similar result holds for derivatives with respect to y. Since both y^n and $y\theta + 1$ are invariant under σ, it follows that

$$(Dh)(x, y) = \zeta(Df)(\zeta x, \zeta y).$$

If $Df = gf$, then

$$(Dh)(x, y) = \zeta g(\zeta x, \zeta y)h(x, y).$$

In particular h is an algebraic solution of D.

Lemma 5.2. *Let $n \geq 2$ be an integer and let $\theta \in \mathbb{Q}[x, y]$ be a homogeneous polynomial of degree $n - 1$ such that:*

(1) $y^n - x\theta$ *is irreducible over* $\mathbb{Q}$*;*
(2) $\theta(1,0) \neq 0$*;*
(3) $(\partial\theta/\partial y)(1,0) \neq 0$.

Then $\mathcal{F}(y^n, y\theta + 1)$ *is a nondicritical foliation of* $\mathbb{P}^2$.

Proof. As we have seen in section 2, $\mathcal{F}(y^n, y\theta + 1)$ is defined by the 1-form

$$\omega = z(y\theta + z^n)dx - zy^n dy + (y^{n+1} - x(y\theta + z^n))dz.$$

Hence its singular set is determined by the equations $z = y(y^n - x\theta) = 0$. Moreover all of its singularities belong to the open set $x \neq 0$. Now $\mathcal{F}(y^n, y\theta+1)$ is represented at $x = 1$ by the vector field

$$(y^{n+1} - y\theta - z^n)\partial/\partial y + zy^n\partial/\partial z.$$

The jacobian of this field at a point with $z = 0$ is

$$\begin{bmatrix} (n+1)y^n - \theta - y\partial\theta/\partial y & 0 \\ 0 & y^n \end{bmatrix}.$$

Since $\theta(1,0) \neq 0$, it follows that the singularity $[1 : 0 : 0]$ is simple, hence nondicritical.

We will show that at a singularity with $y \neq 0$ the ratio of the eigenvalues cannot be a rational number. In particular these singularities are also simple, and the lemma is proved. Suppose, then, by contradiction, that

$$\left(\frac{(n+1)y^n - \theta - y\partial\theta/\partial y}{y^n}\right)(1, a, 0) = q \in \mathbb{Q},$$

where $[1 : a : 0]$ is a singular point of the foliation and $a \neq 0$. Then a is a root of

$$f(y) = ((n+1-q)y^n - \theta - y\partial\theta/\partial y)(1, y, 0),$$

which is a one variable polynomial with rational coefficients. But a is also a root of $g(y) = (y^n - \theta)(1, y)$. Since g is irreducible over $\mathbb{Q}$ by hypothesis, it follows that g divides f. Performing the division we conclude that $((n-q)\theta - y\partial\theta/\partial y)(1, y)$ must be identically zero. There are two cases to consider. First, we might have that $n = q$; but this implies that $(\partial\theta/\partial y)(1, y) = 0$, which has been excluded by hypothesis. On the other hand, if $n \neq q$ then $\theta(1, 0) = 0$, which has also been ruled out by hypothesis.

Corollary 5.3. *Let* p *be a prime integer. If*

$$\theta(x, y) = py^{n-1} + pxy^{n-2} + \cdots + px^{n-2}y + px^{n-1},$$

then $\mathcal{F}(y^n, y\theta + 1)$ *is a nondicritical foliation for all* $n \geq 2$.

Proof. We need only show that θ satisfies the three conditions of lemma 5.2. It is clear that conditions (2) and (3) are satisfied, while condition (1) follows from the fact that $y^n - \theta(1, y)$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion.

Lemma 5.4. *If* $n \geq 2$ *and* $\theta(1,0) \neq 0$*, then* D *has no linear algebraic solutions over* $\mathbb{C}$.

Proof. We argue by contradiction. Suppose that $\lambda = ax + by + c$ is a non-constant solution of D. Equating homogeneous components of the same degree on both sides of $D(\lambda) = g\lambda$, one obtains the system of equations

$$g_{n-1}(ax + by) = ay^n + by\theta \tag{5.1}$$

$$cg_{n-i} + (ax + by)g_{n-i-1} = 0$$

$$cg_0 = b, \tag{5.2}$$

where $1 \leq i \leq n-1$.

It follows from (5.2) that if $c = 0$ then $b = 0$. But this implies that $ay^n = g_{n-1}ax$, which can hold only if $a = 0$. Thus $\lambda = 0$, and we get a contradiction. Therefore we may assume that $c \neq 0$. But this allows us to solve the system by a simple recurrence, which gives $c^n g_{n-1} = (-1)^{n-1}b(ax + by)^{n-1}$. Together with (5.1) this implies that

$$ay^n + by\theta = \frac{(-1)^{n-1}b(ax + by)^n}{c^n}. \tag{5.3}$$

Making $y = 0$ in (5.3), we get that $bax = 0$, which implies that $b = 0$ or $a = 0$. But $b = 0$ implies $a = 0$ by (5.3); contradicting the fact that λ is not constant. On the other hand, if $a = 0$ and $b \neq 0$ we get that $c^n\theta = (-1)^{n-1}b^n y^{n-1}$. Hence $\theta(1, 0) = 0$, which contradicts the hypotheses.

Theorem 5.5. *Let $n \geq 2$ be a prime number and let $\theta \in \mathbb{Q}[x, y]$ be a homogeneous polynomial of degree $n - 1$ such that:*

(1) $y^n - x\theta$ is irreducible over $\mathbb{Q}$;
(2) $\theta(1, 0) \neq 0$;
(3) $(\partial\theta/\partial y)(1, 0) \neq 0$.

Then $D = y^n\partial/\partial x + (y\theta + 1)\partial/\partial y$ has no algebraic solution.

Proof. By [12, proposition 3.3, p. 36], if D has an algebraic solution then it has an algebraic solution in $\mathbb{Q}[x, y]$. Moreover, by lemma 5.2 and corollary 2.5 $\deg f \leq n + 1$. Let $f \in \mathbb{Q}[x, y]$ be an algebraic solution of D, that is *irreducible* as a polynomial over $\mathbb{Q}$. By lemma 5.1 the polynomials $\sigma(f), \sigma^2(f), \ldots, \sigma^{n-1}(f)$ are also algebraic solutions of D. Thus

$$g = f\sigma(f)\sigma^2(f)\cdots\sigma^{n-1}(f)$$

is an algebraic solution of D. Since the foliation of $\mathbb{P}^2$ induced by D is nondicritical by lemma 5.2, it follows that if g is reduced then $\deg(g) \leq n + 1$ by corollary 2.5. But this implies that f has degree 1, and this has been ruled out by lemma 5.4. We are left with the possibility that g is not reduced. But this can happen only if $\sigma^r(f) = cf$, for some $1 \leq r \leq n-1$ and $c \in \mathbb{C}\backslash\{0\}$. Since n is prime, we may assume without loss of generality that $r = 1$. So the previous equality becomes $\sigma(f) = cf$. Decomposing f into homogeneous components, we obtain from $\sigma(f) - cf = 0$ that

$$(\zeta - c)(f_{n+1} + f_1) + (1 - c)(f_n + f_0) + \sum_{i=2}^{n-1}(\zeta^i - c)f_i = 0.$$

Since this identity must hold for all complex numbers x, y, and c is a constant, it follows that

(1) $f = f_k$ for some $0 \leq k \leq n + 1$; or
(2) $f = f_n + f_0$; or

(3) $f = f_{n+1} + f_1$;

where f_{n+1}, f_n, f_1 and f_0 are nonzero.

But if f is a homogeneous polynomial then its linear irreducible factors in $\mathbb{C}[x, y]$ are also solutions of D. However, this has already been ruled out by lemma 5.4. Therefore, we need only consider cases (2) and (3).

First case: $f = f_n + f_0$.

Let g be a polynomial of degree $n-1$ such that $D(f) = gf$. Then

$$Df = D_0 f_n + \partial f_n/\partial y$$

must be equal to

$$g_{n-1} f_n + g_{n-1} f_0 + r f_n + r f_0,$$

where $r = g_{n-2} + \cdots + g_0$. Since Df does not have any term of degree j, for $n \leq j \leq 2n-2$, it follows that $rf_n = 0$. Hence $r = 0$ and $g = g_{n-1}$. But this implies that

$$f_0 g_{n-1} = \partial f_n/\partial y. \tag{5.4}$$

Applying [12, proposition 3.3, p. 36] to $D_0 f_n = f_n g_{n-1}$, we conclude that each irreducible factor of f_n, as a polynomial over $\mathbb{Q}$, must divide $y(y^n - x\theta)$. Since $y^n - x\theta$ is irreducible over $\mathbb{Q}$ by hypothesis, it follows that f_n is either y^n or $y^n - x\theta$. In the former case $D_0(y^n) = ny^n\theta$, so the corresponding g_{n-1} is $n\theta$. Hence, (5.4) becomes

$$nf_0\theta = ny^{n-1}$$

which is impossible because $\theta(1, 0) \neq 0$. On the other hand, if $f_n = y^n - x\theta$, then $g_{n-1} = y\partial\theta/\partial y$. Thus, by (5.4)

$$(f_0 y + x)\partial\theta/\partial y = ny^{n-1}, \tag{5.5}$$

which is a contradiction, since $f_0 y + x$ does not divide y^{n-1}.

Second case: $f = f_{n+1} + f_1$.

Let g be a polynomial of degree $n-1$ such that $D(f) = gf$. Then

$$Df = D_0 f_{n+1} + D_0 f_1 + \partial f_{n+1}/\partial y + \partial f_1/\partial y$$

must be equal to

$$g_{n-1} f_{n+1} + g_{n-1} f_1 + r f_{n+1} + r f_1,$$

where $r = g_{n-2} + \cdots + g_0$. Since Df does not have any term of degree j, for $n+1 \leq j \leq 2n-1$, it follows that $rf_{n+1} = 0$. Hence $r = 0$ and $g = g_{n-1}$. But this implies that $\partial f_1/\partial y = 0$, so that $f_1 = ax$ for some nonzero constant a.

Comparing now terms of degree n on both sides of $Df = gf$, we get

$$ay^n + \partial f_{n+1}/\partial y = axg_{n-1}. \tag{5.6}$$

Applying [12, proposition 3.3, p. 36] to $D_0 f_{n+1} = f_{n+1} g_{n-1}$, we find that there are two possibilities for f_{n+1}, namely y^{n+1} and $y(y^n - x\theta)$. In the first case we get from (5.6) that

$$ay^n + (n+1)y^n = axg_{n-1},$$

which is clearly impossible. Next we must consider what happens when $f_{n+1} = y(y^n - x\theta)$. It then follows from (5.6) that

$$(a + (n+1))y^n = x(\theta + y\partial\theta/\partial y)axg_{n-1}.$$

Therefore $a = -(n+1)$, and

$$(n+1)g_{n-1} = \theta + y\partial\theta/\partial y.$$

However, it follows from $D_0 f_{n+1} = f_{n+1}g_{n-1}$ that $g_{n-1} = \theta + y\partial\theta/\partial y$, which implies that $n = 0$, and gives a contradiction.

6. Comments and problems

We saw in the introduction that Bergman's operator

$$d = \partial/\partial x + (1+xy)\partial/\partial y$$

was the first example of a derivation with respect to which $K[x, y]$ is d-simple. It is not difficult to prove this property by a direct computation. However, this also follows from the following result of A. Shamsuddin.

Theorem 6.1. *Let R be a commutative domain and let $f \in R[y]$ be a polynomial of degree at most 1 in y. Assume that there exists a derivation d of $R[y]$ such that*

(1) $d(R) \subseteq R$;
(2) R is d-simple;
(3) $d(y) = f(y)$;
(4) $d(r) \neq f(r)$ for all $r \in R$.

Then the polynomial ring $R[y]$ is d-simple.

For a proof see [1, theorem 2.3.16, p. 79] or [5, proposition 3.2, p. 410]. It turns out that the method described in this paper applies to Bergman's example. Indeed, a simple computation shows that this holds for a family of derivations to which Bergman's example belongs.

Proposition 6.2. *Let $f \in \mathbb{C}[x, y]$ be a polynomial of degree n. Denote by $\mathcal{F}_g$ the foliation of $\mathbb{P}^2$ that corresponds to $\partial/\partial x + g\partial/\partial y$. Let $g = y\alpha + \beta$, where $\alpha, \beta \in \mathbb{C}[x]$ have degrees $n-1$ and k respectively, and $n - 1 \geq \max\{k-1, 2\}$. The foliation $\mathcal{F}_g$ is non-dicritical at all of its singularities.*

However, if g has degree greater than 1 in y then $\mathcal{F}_g$ has a dicritical singularity at $[0 : 1 : 0]$.

Although there are bounds for the degree (and Castelnuovo-Mumford regularity) of algebraic solutions of one dimensional foliations in higher dimensions, they require the curve to have only ordinary nodes for singularities; see [6]. However, derivations of $\mathbb{C}[x_1, \ldots, x_n]$ without singularities always give rise to foliations of $\mathbb{P}^n$ with degenerate singularities. Thus we cannot apply these bounds. In particular, the results of this paper are not immediately generalizable to higher dimensions.

Despite that, the examples of the previous sections can be used together with 6.1 to construct new examples in higher dimensions. Note that condition (4) of Shamsuddin's theorem is easily satisfied by choosing f with large enough degree.

Finally, the result one would ultimately wish to prove is the following:

Problem 6.3. *Let $n \geq 2$. Is it true that if (a, b) is a generic unimodular row with $\max\{\deg(a), \deg(b)\} \leq n$, then $\mathcal{F}(a, b)$ has no algebraic solutions apart from L_∞?*

However, in order to apply the methods of this paper to problem 6.3 it is necessary to determine the irreducible components of the variety that parametrises unimodular rows up to a a given degree.

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