BOUNDING SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract. We present an algorithmic strategy to compute an upper bound for the degree of the algebraic solutions of non-degenerate polynomial differential equations in dimension two.

1. Introduction

In 1878 Gaston Darboux published a paper [8] on which both Poincaré [21, p. 193] and Painlevé [20, p. 217] bestowed the adjective "magistral". In this paper, Darboux introduced a new method for finding a first integral of a differential equation defined by a homogeneous projective 1-form Ω;see section 2 for definitions. More precisely, Darboux showed that if Ω has degree n and more than $(n+1)(n+2)/2$ distinct algebraic solutions, then it has a first integral. Thus, in order to apply Darboux's method it is enough to find the algebraic solutions of a given polynomial differential equation.

Drawing inspiration from Darboux's work, and from later work by Painlevé and Autonne, Poincaré published in 1891 a paper [21] devoted to the investigation of the rational first integrals of Ω. Both the numerator and the denominator of such an integral must be algebraic solutions of Ω. Poincaré points out that, in principle, this reduces the problem of finding a rational first integral to that of bounding the degrees of the algebraic solutions. At least that is so if, in Poincaré's word, one can "find some way of expressing, in the inequalities, that this integral is irreducible", by which he means that it cannot be written as the composition of a polynomial with a rational first integral.

After lying dormant for some decades, the work of Darboux was reworked in the language of modern algebraic geometry, and vastly generalized, by J.-P. Jouanolou in [14]. As a consequence of [14, Proposition 4.1(ii), p. 126], one has that the degree of the smooth algebraic solutions of Ω cannot be greater than $n+1$. Moreover, it follows from [14, Theoreme 3.3, p. 102] that Ω has infinitely many algebraic solutions if and only if it has a rational first integral. In particular, if Ω does not have a rational first integral, then there is an upper bound for the degree of its algebraic solutions. In the early 1980s, the problem of bounding the degrees of algebraic solutions also appeared in the work of Prelle and Singer on elementary first integrals of differential equations [22, Problem D, p. 227].

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The *Poincaré problem*, as the problem of bounding the algebraic solutions of polynomial vector fields came to be called, was explicitly stated in a paper of D. Cerveau and A. Lins Neto [4, Theorem 1, p. 891]. This paper also contains a generalization of Jouanolou's bound to nodal curves. From the algorithmic point of view, this result has the advantage that there are hypotheses on Ω that force its algebraic solutions to be nodal curves. Similar bounds were obtained by S. Walcher in [24] and M. Carnicer in [3]. All these bounds require Ω to satisfy some extra hypothesis, in addition to the non-existence of a first integral.

In this paper we approach this problem from a different point of view. Instead of giving a formula for the required upper bound, we describe an algorithmic strategy that can be used to find such a bound for a given polynomial 1-form. This strategy is not guaranteed to succeed, and requires the foliation $\mathcal{F}$, induced by Ω on the complex projective plane, to be non-degenerate. However, it produces upper bounds for differential equations that are not covered by any of the previous results; for example, when the foliation has dicritical singularities, see section 5.

The algorithms are based on two index formulae proved by Moulin-Ollagnier in [19], one of which is a version of the well-known Camacho-Sad Index Theorem [2]. The key point of Moulin-Ollagnier's formulae is that the indices may be written as linear combinations, with non-negative integer coefficients, of the characteristic ratios of the foliations at its singularities.

Throughout the paper we assume that the foliation $\mathcal{F}$ is non-degenerate and that it is defined over an *effective field* $K \subset \mathbb{C}$. In other words, K is a field for which the operations of sum, subtraction, multiplication, and division by non-zero elements can be implemented in a computer. Under these hypotheses, the index formulae give rise to a number of systems of diophantine equations of degrees 1 and 2, one of whose variables corresponds to the degree of any algebraic solutions the foliation may have. Thus, whenever all these systems have a finite number of solutions, we get an upper bound on the degree of the algebraic solutions of the foliation. The diophantine equations can be easily determined using the algorithms of section 4, where we also describe the strategy used to compute the desired upper bound. The theorems on which these algorithms are based are proved in section 3. The strategy of section 4 is applied to a number of examples in section 5. Since all our results are stated in terms of holomorphic foliations, we review, in section 2, the concepts and main results of this theory that are used in later sections. All the algorithms described in this paper were implemented for the base field $\mathbb{Q}$ using the computer algebra system Axiom [7]. More details on the implementation can be found in section 4. A file with all the algorithms can be downloaded from `http://www.dcc.ufrj.br/~collier/Folia.html`.

2. Preliminaries

Throughout the paper we denote by $\mathbb{P}^2$ the complex projective plane with homogeneous coordinates x, y and z. A *foliation* $\mathcal{F}$ *of degree* n of $\mathbb{P}^2$ is defined by a homogeneous 1-form

$$\Omega = Adx + Bdy + Cdz, \tag{2.1}$$

where $A, B, C \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of degree $n + 1 \geq 2$ for which $xA + yB + zC = 0$. We say that $\mathcal{F}$ is *defined over a subfield* K *of* $\mathbb{C}$ if $A, B, C \in K[x, y, z]$. The *singular set* $\mathrm{Sing}(\mathcal{F})$ of $\mathcal{F}$ is the set of common zeros of A, B and C in $\mathbb{P}^2$. It follows from Bézout's theorem that $\mathrm{Sing}(\mathcal{F})$ is finite if and only if $\gcd(A, B, C) = 1$. When this is the case, we can assume that, up to a projective transformation, all the singularities of such a foliation belong to the open set $D_+(z)$, of points of $\mathbb{P}^2$ whose z-coordinates are non-zero. Let $C \subset \mathbb{P}^2$ be the set of zeros of a non-constant, homogeneous polynomial $F \in \mathbb{C}[x, y, z]$. The curve C is an *algebraic solution* of $\mathcal{F}$ if it is reduced and $\Omega \wedge dF = F\eta$, for some homogeneous 2-form η with coefficients in $\mathbb{C}[x, y, z]$. In this case we also say that C is invariant under $\mathcal{F}$ or that F is a Darboux polynomial of $\mathcal{F}$. The following two theorems will play a key role in the proofs of the results of the next section. Their proofs can be found in [14, Proposition 4.1, p. 126] and [18, Proposition 3.3, p. 36], respectively.

Theorem 2.1. *Every algebraic solution of a foliation $\mathcal{F}$ of $\mathbb{P}^2$ contains at least one singularity of $\mathcal{F}$.*

Theorem 2.2. *If a foliation of $\mathbb{P}^2$, defined over a subfield K of $\mathbb{C}$, has an algebraic solution, then it has a (possibly different) algebraic solution defined over K.*

The restriction of a foliation $\mathcal{F}$, defined by a homogeneous 1-form Ω of (2.1), to the open set $D_+(z)$ is given by a 1-form

$$\omega = adx + bdy, \tag{2.2}$$

where $a = A(x, y, 1)$ and $b = B(x, y, 1)$. Conversely, if $\pi : D_+(z) \to \mathbb{C}^2$ is the map defined by $\pi([x : y : z]) = (x/z, y/z)$, then $\Omega = z^k \pi^*(\omega)$, where k is the smallest positive integer for which the resulting homogeneous form has no poles. Thus, $\mathcal{F}$ is completely defined by ω. Similarly, a square-free, non-constant, homogeneous polynomial $F \in \mathbb{C}[x, y, z]$ defines an algebraic solution of $\mathcal{F}$, if and only if there exists $g \in \mathbb{C}[x, y]$ such that

$$\omega \wedge df = f \cdot gdx \wedge dy, \tag{2.3}$$

where $f = F(x, y, 1)$ is the dehomogenization of F with respect to z. It is also easy to show that, if all the singularities of $\mathcal{F}$ are contained in $D_+(z)$, then

$$\deg(\mathcal{F}) = \max\{\deg(a), \deg(b)\} - 1 = n.$$

Suppose that the foliation $\mathcal{F}$, defined by (2.2), has finitely many singularities, all of them contained in $D_+(z)$, and let

$$J_{\mathcal{F}} = \begin{bmatrix} \partial b/\partial x & \partial b/\partial y \\ -\partial a/\partial x & -\partial a/\partial y \end{bmatrix}.$$

The *jacobian* of $\mathcal{F}$ at a singularity p is the matrix $J_{\mathcal{F}}(p)$ obtained by specializing x and y to the corresponding coordinates of p. When $\det(J_{\mathcal{F}}(p))$ is non-zero, $\mathcal{F}$ is said to be *non-degenerate at* p. A foliation is *non-degenerate* if it is non-degenerate at all of its singular points. The following characterization is very helpful in checking that a given foliation is non-degenerate; see [23] for details.

Theorem 2.3. *A foliation of degree n in $\mathbb{P}^2$ is non-degenerate if and only if it has $n^2 + n + 1$ distinct singularities.*

Next we define a number of indices for foliations of $\mathbb{P}^2$. We will assume, from now on, that $\mathcal{F}$ is a non-degenerate foliation defined on $D_+(z)$ by a 1-form ω and that $\mathrm{Sing}(\mathcal{F}) \subset D_+(z)$. Under these hypotheses, the *Baum-Bott index* of $\mathcal{F}$ at a singularity p is

$$BB(\mathcal{F}, p) = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1 \lambda_2} = \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + 2, \tag{2.4}$$

where λ_1 and λ_2 are the eigenvalues of $J_{\mathcal{F}}(p)$. Moreover, by the Baum-Bott Theorem,

$$\sum_{p\in \mathrm{Sing}(\mathcal{F})} BB(\mathcal{F}, p) = (n+2)^2. \tag{2.5}$$

The other two indices are defined with respect to an algebraic solution C of $\mathcal{F}$. Let $F \in \mathbb{C}[x, y, z]$ be the square-free homogeneous polynomial whose vanishing defines C, and let $p \in \mathrm{Sing}(\mathcal{F}) \cap C$. Applying a projective transformation, if necessary, we can assume that $p = 0$ and that the linear part of ω has the form

$$\omega_1 = \lambda_2 y dx - (\lambda_1 x + \alpha y)dy,$$

where λ_1 and λ_2 are the eigenvalues of $J_{\mathcal{F}}(p)$ and α is a complex number. Let k be the *order* of $f = F(x, y, 1)$ at p; that is, k is the smallest non-negative integer for which the Taylor polynomial of f at p has a non-zero monomial. It follows from (2.3), that there exists $(\nu, \mu) \in \mathbb{N}^2$ such that

$$\nu + \mu = k \ \text{ and } \ \nu\lambda_1 + \mu\lambda_2 = g(p).$$

A tuple that satisfies these conditions will be called an *LL-pair* of $\mathcal{F}$ at p. Note that, although there may be several LL-pairs for a given singularity p, the corresponding linear combinations of the eigenvalues are always equal to $g(p)$. We will call

$$\rho_1 = \lambda_1/\lambda_2 \ \text{ and } \ \rho_2 = 1/\rho_1$$

the *characteristic ratios* of $\mathcal{F}$ at p. In [19, equation 20, p. 259] Moulin-Ollagnier defined the following indices of $\mathcal{F}$ at p,

$$\iota_1(p, C, \mathcal{F}) = \frac{(\nu\lambda_1 + \mu\lambda_2)(\lambda_1 + \lambda_2)}{\lambda_1\lambda_2} = \nu\rho_1 + \mu\rho_2 + (\nu + \mu), \tag{2.6}$$

$$\iota_2(p, C, \mathcal{F}) = \frac{(\nu\lambda_1 + \mu\lambda_2)^2}{\lambda_1\lambda_2} = \nu^2\rho_1 + \mu^2\rho_2 + 2\nu\mu, \tag{2.7}$$

where (ν, μ) is an LL-pair of $\mathcal{F}$ at p. The subscripts 1 and 2, in the notation of the indices, correspond to their degrees as polynomials in μ and ν. The index ι_2 is a version of the well-known Camacho-Sad index; see [2]. There is a very important special case in which these indices are easy to compute.

Proposition 2.4. *Let p be a singularity and let C be an algebraic solution of a non-degenerate holomorphic foliation $\mathcal{F}$ of $\mathbb{P}^2$. If $p \in C$ and the characteristic ratios of $\mathcal{F}$ at p do not belong to $\mathbb{Q}^+$ then the LL-pair of C at p must be equal to* $(1, 0)$, $(0, 1)$ *or* $(1, 1)$.

Proof. Let $\mathcal{F}$ be defined by the 1-form ω in $\mathrm{D}_+(z)$. Without loss of generality we may assume that $p \in \mathrm{D}_+(z)$ is the origin and that the linear part of ω at p is $\lambda_2 y dx - \lambda_1 x dy$. Note that, since the characteristic ratios cannot be equal to one, the jacobian must be diagonalizable. Let $h \in \mathbb{C}\{x, y\}$ be a germ of smooth holomorphic function at p that satisfies

$$\omega \wedge dh = h \cdot g dx \wedge dy, \tag{2.8}$$

for some $g \in \mathbb{C}\{x, y\}$. Since we are assuming that h is smooth, its linear homogeneous component has the form $\alpha x + \beta y$, where $\alpha, \beta \in \mathbb{C}$ are not both zero. Thus, we get from (2.8) that

$$\lambda_1 \alpha x + \lambda_2 \beta y = g(p)(\alpha x + \beta y).$$

But this implies that either $\alpha = 0$ or $\beta = 0$ for, otherwise, we would have $g(p) = \lambda_1 = \lambda_2$, which contradicts our assumption that $\rho_1 \notin \mathbb{Q}^+$. In particular, if C is smooth, then its LL-pair at p must be either $(1, 0)$ or $(0, 1)$. Suppose now that C is singular at p. If $\rho_1 \notin \mathbb{Q}$, [1, Proposition 2, p. 230], or $\rho_1 \notin \mathbb{R}^+$, [17, Proposition 2.5, p. 656], then C has two transverse smooth analytic branches at p. But if C is given by the vanishing of a square-free polynomial $f \in \mathbb{C}[x, y]$, then f also has a square-free factorization in the ring $\mathbb{C}\{x, y\}$; see [16, Theorem 16.5, p. 163]. Thus, $f = h_1 h_2$, where h_1 and h_2 are smooth at p. Applying the first part of the proof to h_1 and h_2 and taking into account that the branches are transversal at p, we conclude that the LL-pair of C at p must be $(1, 1)$. □

Corollary 2.5. *Let p be a singularity and let C be an algebraic solution of a holomorphic foliation $\mathcal{F}$ of $\mathbb{P}^2$. If $p \in C$ and the characteristic ratios ρ_1 and ρ_2 of $\mathcal{F}$ at p do not belong to $\mathbb{Q}^+ \cup \{0\}$, then, for $i = 1, 2$,*

$$\iota_1(p, C, \mathcal{F}) = \begin{cases} \iota_2(p, C, \mathcal{F}) + 1 = \rho_i + 1 & \text{when } C \text{ is smooth at } p; \\ \iota_2(p, C, \mathcal{F}) = \rho_1 + \rho_2 + 2 & \text{when } C \text{ is nodal at } p. \end{cases}$$

If $S \subset \mathrm{Sing}(\mathcal{F})$ then, for $j = 1, 2$, we denote by $\iota_j(S, C, \mathcal{F})$ the sum of the ι_j-indices of $\mathcal{F}$ at the points of S. The second equation of the next theorem is a version of the well-known Camacho-Sad Index Theorem, while the first is related to a result of Brunella. See [19, p. 259] for a proof of both results.

Theorem 2.6. *Let $\mathcal{F}$ be a non-degenerate foliation of degree n of $\mathbb{P}^2$ and let C be an algebraic solution of degree m of $\mathcal{F}$. If $S = C \cap \mathrm{Sing}(\mathcal{F})$, then*

$$\iota_1(S, C, \mathcal{F}) = m(n + 2) \quad \text{and} \quad \iota_2(S, C, \mathcal{F}) = m^2.$$

In order to use the formulae of Theorem 2.6 we must be able to compute the sums of indices on their left hand side. We will do this using the Gröbner bases of certain zero dimensional ideals. More precisely, let I be an ideal of a polynomial ring $K[x_1, \dots, x_n]$ whose monomials are ordered lexicographically with $x_1 < \cdots < x_n$. We will also assume that I is in *general position* with respect to x_1, which means that, all the points in the zero set of I have distinct first coordinates. Under these hypotheses we can use the Shape Lemma [15, Theorem 3.7.25, p. 257], to conclude that $\sqrt{I}$ is an intersection of ideals of the form $(g_1, x_2 - g_2, \dots, x_n - g_n)$, where $g_1, \dots, g_n \in K[x_1]$ and g_1 is irreducible over K. This decomposition can be computed using the AXIOM function `groebnerFactorize`, which implements the factorizing Gröbner basis algorithm, see [11, Exercise 4.5.4], for example.

3. Sums of indices

Taking into account the results of section 2, we will assume, throughout this section, that $\mathcal{F}$ is a foliation of $\mathbb{P}^2$ defined by a 1-form $\omega = adx + bdy$, such that

(1) $a, b \in K[x, y]$, where $K \subset \mathbb{C}$ is an effective field;
(2) $n = \max\{\deg(a), \deg(b)\} - 1 \geq 2$;
(3) ω has $n^2 + n + 1$ singularities, no two of which have the same x-coordinate.

Since $\mathcal{F}$ is now fixed, we often drop it from the notation altogether. Recall that (2) and (3) imply that $\mathcal{F}$ is non-degenerate.

We begin by defining two polynomials that will be required in the computation of the sums of indices with which we will be concerned. Let $p_0 = (x_0, y_0)$ be a singular point of $\mathcal{F}$. It follows from (2.4) that if

$$bb(x, y, u) = \det(J_{\mathcal{F}})u - \text{trace}(J_{\mathcal{F}})^2 \in K[x, y, u],$$

then the root of the linear polynomial $bb_{p_0}(u) = bb(x_0, y_0, u)$ is the Baum-Bott index of $\mathcal{F}$ at p_0. The same equation also implies that the characteristic ratios of $\mathcal{F}$ at p_0 are the roots of the quadratic polynomial $\chi_{p_0}(t) = \chi_{\mathcal{F}}(x_0, y_0, t)$, where

$$\chi_{\mathcal{F}}(x, y, t) = \det(J_{\mathcal{F}})t^2 + (2\det(J_{\mathcal{F}}) - \text{trace}(J_{\mathcal{F}})^2)t + \det(J_{\mathcal{F}}), \qquad (3.1)$$

is a polynomial with coefficients in K.

It follows from (3) and Theorem 2.3 that (a, b) is a radical ideal of dimension zero of $K[x, y]$, in general position with respect to x. Thus, by the Shape Lemma [15, Theorem 3.7.25, p.257], the factorizing Gröbner basis algorithm applied to (a, b) (with respect to the lexicographical order with $x < y$) returns polynomials $f_j, g_j \in K[x, y]$, such that the f_j is irreducible and $\text{Sing}(\omega)$ is the union of the sets

$$S_j = \{(a, g_j(a)) \mid a \in \overline{K} \text{ and } \quad f_j(a) = 0\}.$$

for $1 \le j \le k$. We call the S_j the *singular blocks* of ω. Denoting by $\deg(S_j)$ the degree of the polynomial f_j, and taking into account that the ideal (a, b) is in general position with respect to x, it follows that

$$\sum_{j=1}^{k} \deg(S_j) = n^2 + n + 1.$$

Now let ϕ_j be a generator of the radical of $(f_j, y - g_j, \chi_{\mathcal{F}}) \cap K[t]$. When the argument is limited to just one singular block S, we will replace the index j by S, or omit it altogether.

The absolute Galois group G of K acts on the block S defined by the ideal $(f, y - g)$ by

$$\sigma \cdot (x_0, g(x_0)) = (\sigma(x_0), g(\sigma(x_0))),$$

where $\sigma \in G$ and $f(x_0) = 0$. Since the Galois group of f over K is a homomorphic image of G [13, Theorem 8.5, p. 470] and f is irreducible, it follows that G acts transitively on S. This action extends to the set of Baum-Bott residues of ω at the points of the block S because $bb(x, g(x), t) \in K[x, t]$ is a polynomial of degree one in t. The next result is concerned with the action of G on set $\text{Ch}(S)$ of characteristic ratios of $\mathcal{F}$ at the points of S. Let $\kappa(S) = 2\deg(f)/\deg(\phi_S)$.

Theorem 3.1. *If S is a block of singularities of $\mathcal{F}$, then ϕ_S has at most two distinct irreducible factors over K. Moreover, each one of its roots is the characteristic ratio of $\kappa(S)$ points of S.*

Proof. Let S be the block defined by the prime ideal $(f, y - g)$ of $K[x, y]$, where $f, g \in K[x]$, and consider the points $p_0 = (x_0, g(x_0))$ and $p_1 = (x_1, g(x_1))$ of S. Since f is irreducible and x_0 and x_1 are two of its roots, there exists $\tau \in G$ that maps x_0 to x_1. Hence, $\tau(p_0) = p_1$ and $\chi_{p_0}^{\tau} = \chi_{p_1}$. The characteristic ratios ρ_0 and $1/\rho_0$ of $\mathcal{F}$ at $p_0 = (x_0, g(x_0)) \in S$ are the roots of the polynomial $\chi_{p_0}(t)$, which has degree 2 in t.

Suppose, first, that $\chi_{p_0}(t)$ is irreducible over $K[x_0]$. It follows from $\chi_{p_1} = \chi_{p_0}^{\tau}$ that χ_{p_1} is irreducible over $K[x_1] = \tau(K[x_0])$, so that both p_0 and p_1 have two distinct characteristic ratios each. Hence, by [6, Proposition 2, p. 183], there exist elements of G that map ρ_0 into each one of the characteristic ratios at p_1. In particular, G acts transitively on $\mathrm{Ch}(S)$, which implies that ϕ_S is irreducible over K by [6, Theorem 1, p. 202].

Assume, now, that x_1 is conjugate to x_0 over $K[\rho_0]$. Then, there exists $\sigma \in G$ such that $\sigma(x_0) = x_1$ and $\sigma(\rho_0) = \rho_0$. But this implies that $\sigma(p_0) = p_1$ and that

$$0 = \sigma(\chi_{p_0}(\rho_0)) = \chi_{\sigma(p_0)}(\sigma(\rho_0)) = \chi_{p_1}(\rho_0).$$

Hence, ρ_0 is also a characteristic ratio at p_1. However, x_0 has

$$[K[x_0, \rho_0] : K[\rho_0]] = \frac{[K[x_0, \rho_0] : K]}{[K[\rho_0] : K]} = \frac{2\deg(f)}{\deg(\phi_S)} = \kappa(S)$$

conjugates over $K[\rho_0]$, which proves the theorem in this case. Note that this includes the case in which $\phi_S = t \pm 1$.

When $\chi_{p_0}(t)$ is reducible over $K[x_0]$, there exists a polynomial $\theta \in K[x]$ such that $\rho_0 = \theta(x_0)$. Thus,

$$\tau(\rho_0) = \tau(\theta(x_0)) = \theta(\tau(x_0)) = \theta(x_1).$$

Since the same holds for $1/\rho_0$ and ϕ_S is reducible, it follows that G has exactly two orbits in $\mathrm{Ch}(S)$. Moreover, if ψ_1 is the minimal polynomial of ρ_0 over K, then the minimal polynomial of $1/\rho_0$ over K is

$$\psi_2 = t^{\deg(\psi_1)}\psi_1(1/t),$$

and $\phi_S = \psi_1 \cdot \psi_2$. In particular, $\deg(\psi_1) = \deg(\psi_2)$, so that ρ_0 is the characteristic ratio of

$$[K[x_0] : K[\rho_0]] = \frac{\deg(f)}{\deg(\psi_1)} = \frac{2\deg(f)}{\deg(\phi_S)} = \kappa(S)$$

elements of S, because

$$\deg(\phi_S) = \deg(\psi_1) + \deg(\psi_2) = 2\deg(\psi_1);$$

which completes the proof of the theorem. □

The following result is an immediate consequence of the proof of Theorem 3.1.

Corollary 3.2. *If S is a block of singularities of $\mathcal{F}$, then one of the following holds:*

1. $\mathrm{Ch}(S) = \{1\}$ *or* $\mathrm{Ch}(S) = \{-1\}$*;*
2. *G acts transitively on* $\mathrm{Ch}(S)$*;*
3. *G has two orbits in* $\mathrm{Ch}(S)$*, each one of which corresponds to the roots of one of the factors of ϕ_S over K.*

Next we must consider what the action of G does to an algebraic solution of $\mathcal{F}$. But first we need a definition. The *minimal homogeneous component* of a polynomial $h \in K[x, y]$ at a point $p \in \mathbb{C}^2$ is the homogeneous component of smallest degree of the Taylor expansion of h at p.

Lemma 3.3. *Let C be an algebraic solution (defined over K) and S a singular block of $\mathcal{F}$ whose intersection with C is non-empty. It follows that:*

1. *$S \subset C$ and C has the same LL-pairs at all the points of S;*
2. *if* $\mathrm{Sing}(C) \cap S \neq \emptyset$ *then* $S \subset \mathrm{Sing}(C)$.

Proof. Let C be the curve defined by the vanishing of $h \in K[x, y]$ and assume that it has non-empty intersection with S. Then, for all $\sigma \in G$ and $p \in S \cap C$,

$$h(\sigma(p)) = \sigma(h(p)) = 0.$$

Taking into account that G acts transitively on S, we get the first part of (1). Similarly, if $p \in \operatorname{Sing}(C) \cap S$ then

$$\nabla h(\sigma(p)) = \sigma(\nabla h(p)) = 0,$$

and (2) follows from the transitivity of the action of G on S. Assume now, that the block S is defined by the ideal $(f, y-g)$ of $K[x,y]$, where $f, g \in K[x]$. Given a point $p = (x_0, g(x_0)) \in S \cap C$, the minimal homogeneous component of h at p can be written in the form

$$h_r = \sum_{i=0}^{r} c_i (x - x_0)^i (y - g(x_0))^{r-i},$$

where $c_0, \dots, c_r \in K[x_0]$. If λ_1 and λ_2 are the eigenvalues of $J_{\mathcal{F}}(p)$, then it follows from $\omega \wedge df = fg dx \wedge dy$ and the definition of LL-pairs that

$$i\lambda_1 + (r-i)\lambda_2 = g(p),$$

for some $0 \leq i \leq r$. Given σ in the absolute Galois group of K, the minimal homogeneous component of h at $\sigma(p_0)$ is

$$(h_r)^\sigma = \sum_{i=0}^{r} \sigma(c_i)(x - \sigma(x_0))^i (y - g(\sigma(x_0)))^{r-i}.$$

Moreover, $\sigma(J_{\mathcal{F}}(p)) = J_{\mathcal{F}}(\sigma(p))$ implies that the eigenvalues of this last matrix are $\sigma(\lambda_1)$ and $\sigma(\lambda_2)$. Hence,

$$i\sigma(\lambda_1) + (r-i)\sigma(\lambda_2) = \sigma(g(p)) = g(\sigma(p)),$$

so that $(i, r-i)$ is also an LL-pair for C at $\sigma(p)$. Since G acts transitively on S, it follows that h must have the same LL-pairs at all the points of S. □

Combined with Theorem 3.1 and Corollary 3.2, this lemma gives a method for computing the sum of ι_2-indices of C at the singularities of S. Given a polynomial g in one variable, we will denote by $s(g)$ the sum of the roots of g.

Proposition 3.4. *Let S be a singular block of $\mathcal{F}$ and let C be an algebraic solution of $\mathcal{F}$, defined over K. If $S \subset C$ and ϕ_S does not have a positive rational root then there exists a factor ψ of ϕ_S such that*

$$\iota_2(S, C) = \begin{cases} \kappa(S)s(\psi), & \text{if } \psi \neq \phi_S \text{ is irreducible;} \\ \kappa(S)s(\psi) + 2|S|, & \text{otherwise.} \end{cases}$$

Proof. Let $\rho_1(p)$ and $\rho_2(p)$ be the characteristic ratios of $\mathcal{F}$ at $p \in S$. By Lemma 3.3, all the elements of a block have the same LL-pairs. Thus, by (2.7), there exists $(\nu, \mu) \in \mathbb{N}^2$ such that

$$\iota_2(p, C) = \nu\rho_1(p) + \mu\rho_2(p) + 2\nu\mu$$

for all $p \in S$. Therefore,

$$\iota_2(S, C) = \nu \sum_{p \in S} \rho_1(p) + \mu \sum_{p \in S} \rho_2(p) + 2|S|\nu\mu. \tag{3.2}$$

Since we are assuming that ϕ_S does not have a rational root, it follows from Proposition 2.4 that either, (1) C has a node and LL-pair $(1, 1)$ at every point of S, or

(2) C is smooth with the same LL-pair, either $(1,0)$ or $(0,1)$, at every point of S. In case (1), equation (3.2) becomes

$$\iota_2(S,C) = \sum_{p\in S}\rho_1(p) + \sum_{p\in S}\rho_2(p) + 2|S|.$$

Thus, by Theorem 3.1,

$$\iota_2(S,C) = \kappa(S)s(\phi_S) + 2|S|.$$

Turning now to case (2), we assume, without loss of generality that $\mu = 1$ and $\nu = 0$. Taking this into (3.2), we get

$$\iota_2(S,C) = \sum_{p\in S}\rho_1(p).$$

Hence, in this case, ϕ_S must have two proper factors and the numbers $\rho_1(p)$ for $p \in S$ are the roots of one of these factors, which we denote by ψ_S. Therefore, by Theorem 3.1,

$$\iota_2(S,C) = \kappa(S)s(\psi_S),$$

in this case, completing the proof of the proposition. □

Replacing equation (2.7) by (2.6) in the proof of Proposition 3.4, we obtain a similar formula for the sums of ι_1-indices at the points of S.

Proposition 3.5. *Let S be a singular block of $\mathcal{F}$ and let C be an algebraic solution of $\mathcal{F}$, defined over K. If $S \subset C$ and ϕ_S does not have a positive rational root then there exists a factor ψ of ϕ_S such that*

$$\iota_1(S,C) = \begin{cases} \kappa(\psi)s(\psi) + |S|, & \text{if } \psi \neq \phi_S \text{ is irreducible;} \\ \kappa(\psi)s(\psi) + 2|S|, & \text{otherwise.} \end{cases}$$

4. Computing the degrees

We describe, in this section, algorithms that can be used to determine an upper bound on the degree of an algebraic solution of a given foliations of $\mathbb{P}^2$. As we shall see, these algorithms often return more information on the algebraic solutions than the upper bound itself. All the algorithms were implemented, over the effective field $\mathbb{Q}$, in the language of the computer algebra system Axiom .

Throughout the section we will assume that $\mathcal{F}$ is a non-degenerate foliation, defined over an effective field K, which has no singularities at the line $z = 0$. Recall that a block of singularities S of $\mathcal{F}$ is defined by a prime ideal $(f_S, y - g_S)$ of $K[x,y]$, where $f_S, g_S \in K[x]$, and that ϕ_S is the generator of the radical of $(f_S, y - g_S, \chi_{\mathcal{F}}) \cap K[t]$.

We begin by collating the results of the previous section into an algorithm. Of course those results cannot be used directly, because we aim to determine the possible degrees of algebraic solutions of $\mathcal{F}$ before we find the curves themselves. Thus, the ι_1- and ι_2-indices of these curves are unknown, unless the polynomial ϕ_S, of a given block S, does not have positive rational roots. And even in this case there are three possible values for each index, when ϕ_S is not irreducible. To get around this problem, our first algorithm returns the sets of all possible sums of ι_1-indices, and *all possible sums* of ι_2-indices, of an algebraic solution of $\mathcal{F}$ at the singularities of S. Note that, if ϕ_S has positive rational roots, these sets contain, not numbers, but expressions on the variables μ_S and ν_S that stand for the LL-pairs of a possible algebraic solution of $\mathcal{F}$.

Algorithm 4.1 (`sumsInBlock`)**.** *Given a non-degenerate foliation $\mathcal{F}$, defined over K and with no singularities at $z = 0$, and a block S of singularities of $\mathcal{F}$ defined by an ideal $(f, y - g)$ with $f, g \in K[x]$, the algorithm returns the set of pairs whose entries are the expressions that describe, respectively, the sum of ι_1-indices and of ι_2-indices of all possible algebraic solutions of $\mathcal{F}$ over the singularities of S.*

Step 1: *compute the generator ϕ_S of the ideal $\sqrt{(f, y - g, \chi_{\mathcal{F}}) \cap K[t]}$;*

Step 2: *if ϕ_S has (possibly equal) positive rational roots ρ_1 and ρ_2, return*

$$\left\{[deg(f)(\rho_1\nu_S + \rho_2\mu_S + \nu_S + \mu_S),\ deg(f)(\rho_1\nu_S^2 + \rho_2\mu_S^2 + 2\nu_S\mu_S)]\right\}.$$

Step 3: *if ϕ_S is irreducible with no positive rational roots, return*

$$\{[\kappa(S)s(\phi_S) + 2\deg(f),\ \kappa(S)s(\phi_S) + 2\deg(f)]\};$$

Step 4: *if $\phi_S = \psi_1\psi_2$ and the ψs are irreducible with no positive rational roots, return the set whose elements are*

$$[\kappa(S)s(\phi_S) + 2\deg(f),\ \kappa(S)s(\phi_S) + 2\deg(f)]$$

and the pairs

$$[\kappa(S)s(\psi_i), \kappa(S)s(\psi_i) + \deg(f)], \quad \textit{for} \quad i = 1, 2.$$

Proof. The required formulae, when the characteristic ratios are not positive fractions, were proved in Propositions 3.4 and 3.5. Thus, we may assume, for the rest of the proof, that $\rho_1(p_0) \in \mathbb{Q}^+$, for some $p_0 \in S$. Let C be an algebraic solution of $\mathcal{F}$, defined over K, and let $p \in S \cap C$. Choose $\sigma_p \in G$ such that $\sigma_p(p_0) = p$. It follows from $\mathbb{Q}^+ \subset K$ that

$$\rho_1(p) = \sigma_p(\rho_1(p_0)) = \rho_1(p_0).$$

Moreover, by Lemma 3.3, all the elements of a block have the same LL-pairs. Thus, by (2.6), there exists $(\nu, \mu) \in \mathbb{N}^2$ such that

$$\iota_2(p, C, \mathcal{F}) = |S|(\nu\rho_1(p_0) + \mu\rho_2(p_0) + 2\nu\mu).$$

Since $|S| = \deg(f)$, this gives the second entry of the pair of step 2. Applying a similar argument to (2.7) we get the first entry of the pair. ☐

In order to simplify the algorithms we will assume, from now on, that $K = \mathbb{Q}$; otherwise, we would have had to introduce a basis of K over $\mathbb{Q}$ in order to extract the required diophantine equations. By Theorem 2.1, any algebraic solution C of a foliation $\mathcal{F}$ determines (1) the set of blocks whose singularities are contained in C and (2) the sum of the indices at each of these blocks. Conversely, if we know the sums of indices for each one of the blocks of singularities contained in a given algebraic solution of $\mathcal{F}$, we can determine the degree of this algebraic solution using Theorem 2.6.

This suggests the following strategy for determining the degree of an algebraic solution of $\mathcal{F}$. Suppose first that $\mathcal{F}$ does not have positive rational characteristic ratios. By Corollary 2.5, there are only three choices for the sums of indices at each block of singularities of $\mathcal{F}$. Thus, the set of all possible sums of these choices, one for each singular block, includes those that come from the algebraic solutions of $\mathcal{F}$. The possible degrees can then be determined using Theorem 2.6, as above.

Of course not all positive integers obtained in this way are necessarily the degree of an algebraic solution.

Suppose, now, that the characteristic ratios of some of the blocks of singularities of $\mathcal{F}$ are positive rational numbers and let S be one of these blocks. Applying the above algorithm to S, we get expressions in the variables μ_S and ν_S, that represent the entries of the LL-pair of an algebraic solution of $\mathcal{F}$ at S. Hence, if we proceed as in the previous paragraph, we end up not with a set of pairs of numbers, but with a set of pairs of diophantine equations in the μs and νs; one pair of variables for each block whose characteristic ratios belong to $\mathbb{Q}^+$. It turns out that, in several interesting cases, these equations have finitely many non-negative integer solutions, allowing us to find all positive integers that are viable degrees of algebraic solutions of $\mathcal{F}$. Before describing some of the strategies that can be used to bound the solutions of these diophantine equations, we give a detailed version of the algorithm that finds the equations themselves.

Algorithm 4.2 (`indexSums`)**.** *Given a non-degenerate foliation $\mathcal{F}$ of $\mathbb{P}^2$, defined over $\mathbb{Q}$ and with no singularities at $z = 0$, the algorithm returns the sets of all possible sums of ι_1- and ι_2-indices of an algebraic solution of $\mathcal{F}$.*

Step 1: *inicialize two empty sets $\mathcal{I}$ and $\mathcal{S}$;*

Step 2: *find the set $\mathcal{B}$ of singular blocks of $\mathcal{F}$ using the factorizing Gröbner basis algorithm;*

Step 3: *for each block $S \in \mathcal{B}$ add* `sumsInBlock`$(\mathcal{F}, \mathtt{S})$ *to $\mathcal{I}$;*

Step 4: *for each subset of $\mathcal{I}$ and each pair from each element of the subset:*
- *compute the sum σ_j of each of the jth entries of each pair, for $j = 1, 2$;*
- *let eq_1 and eq_2 be the polynomials with integer coefficients obtained by clearing the denominators of $(d+2)m - \sigma_1$ and $m^2 - \sigma_2$;*
- *add $[eq_1, eq_2]$ to $\mathcal{S}$;*

Step 5: *return $\mathcal{S}$.*

Let $\mathcal{D}$ be the set of pairs of diophantine equations returned by `indexSums` when it is applied to $\mathcal{F}$. Given a polynomial h, we will denote by $\mathrm{var}(h)$ the set of those variables that appear in a non-zero summand of h.

Choose a monomial order for which m is greater than all the μs and νs and let $\mathcal{G}$ be the set of all the Gröbner bases obtained applying the Gröbner factorization algorithm to each element of $\mathcal{D}$. Each $g \in \mathcal{G}$ is subject to the following two-step analysis: If g contains a non-constant polynomial in $\mathbb{Z}[m]$, then m can be easily computed. This happens, for example, when none of the characteristic ratios of $\mathcal{F}$ belong to $\mathbb{Q}^+$. Otherwise, g will have polynomials $c \cdot m - \ell$ and q, such that

- $c \in \mathbb{Z}$;
- ℓ and q are polynomials with integer coefficients;
- $\deg(\ell) = 1$ and $\deg(q) = 2$;
- $m \notin \mathrm{var}(q) \cup \mathrm{var}(\ell)$.

Let β be the orthonormal basis of principal directions of the homogeneous quadratic component of q and denote by $\mathfrak{q}$ and $\mathfrak{m}$ the polynomials obtained writing q and ℓ/c in the coordinates of β. Assume that

(1) the Gram matrix of $\mathfrak{q}$, with respect to the variables in $\mathrm{var}(\mathfrak{q})$, is positive definite;

(2) $\text{var}(\mathfrak{m}) \subset \text{var}(\mathfrak{q})$.

By (1) the coordinates of the points of $\mathfrak{q}$ corresponding to the variables in $\text{var}(\mathfrak{q})$ are bounded from above. Since, by (2), $\mathfrak{m}$ depends only on the variables that appear in $\mathfrak{q}$, it follows that m can also be bounded in this case. Of course this approach fails if either (1) or (2) does not hold.

Besides `groebnerFactorize`, the only major function from the Axiom library required in performing these calculations is `orthonormalBasis`, which computes an orthonormal basis of eigenvectors of a symmetric matrix. This last function imposes severe restrictions on our implementation of the algorithms because it fails if there is no explicit expression for the roots of the characteristic polynomial of its input. In principle this could be circumvented using floating point approximations, but that would require a careful analysis of the error propagation.

Once an upper bound on the degree of the algebraic solutions of $\mathcal{F}$ has been found, it can be used to determine the possible LL-pairs at the singularities of $\mathcal{F}$ whose characteristic ratios are positive rational numbers. In order to do this it is enough to substitute the values of m in the pairs of index equations and compute their solutions. Since the sum s of the entries of the LL-pair of a given singularity p of $\mathcal{F}$ is equal to the order of the algebraic solution at p, we must have $s \leq m$. Hence, once m is known, the entries of all feasible LL-pairs can also be found, which speeds up the task of computing the algebraic solutions of $\mathcal{F}$.

The computations required to implement the strategy of the last paragraph can be performed with the help of the Axiom function `dioSolve`. Given a linear diophantine equation, this function returns a set of minimal non-homogeneous solutions to the equation, together with a basis of the solutions of the corresponding homogeneous equation. Any solution of the non-homogeneous equations can then be obtained adding to a minimal solution a linear combination, with non-negative integer coefficients, of the homogeneous solutions. For more details see [5] and [9, pp. 277–281].

5. Examples

In this section we apply the strategy of section 4 to analyse a number of examples of foliations of degrees two and three, defined over $\mathbb{Q}$. The limitation on the degrees has as much to do with the difficulty of producing examples of larger degree, as it does with the cost of the computations. The most conspicuously missing example is that of a non-degenerate foliation, with finitely many algebraic solutions, for which the strategy breaks down.

The six examples in this section, which we will denote by $\mathcal{F}_i$ for $1 \leq i \leq 6$, are defined by 1-forms $\omega_i = a_i dx + b_i dy$, whose coefficients $a_i, b_i \in \mathbb{Q}[x, y]$ will be given explicitly.

5.1. **Example.** We begin with Jouanolou's classic example of a foliation without algebraic solutions. In degree 4 it is defined by the 1-form

$$\omega_1 = (x^4 - y^5)dx - (1 - xy^4)dy.$$

Applying `indexSums` to ω_1 we get 16 systems of two equations on the degree m of the curve alone. The only two of systems with integer solutions give $m = 0$ and $m = 6$. Thus, if ω_1 had an algebraic solution, it would have to be of degree 6. However, this can only happen when the algebraic solution has a node at each one of the 21 singularities of $\mathcal{F}$, which contradicts [10, Theorem 2, section 4, chapter 5].

5.2. **Example.** A more interesting example is the foliation $\mathcal{F}_2$, of degree 3, defined by

$$\begin{aligned} a_2 &= -hy - 4y^3 - 8xy^2 + (-4x^2 - 32)y - 32x \\ b_2 &= hx + 64y^3 + (189x - 24)y^2 + (104x^2 - 24x - 16)y + 6x^3 - 32x. \end{aligned}$$

where $h = 168y^3 + 342xy^2 + 180x^2y + 6x^3$. Applying `indexSums` to $\mathcal{F}_2$ we get 36 pairs of diophantine equations in 3 variables; namely, the degree of the curve and its LL-pairs at the origin. Applying the Gröbner factorization algorithm to these equations, we get one system of equations for each of the 36 pairs, each one of which contains an equation involving only the degree of the curve. Of these equations, only the one corresponding to the system $m = 3$, with $\mu = \nu$ at the origin, has an integral solution. Thus, the homogeneous component of minimal degree of an algebraic solution of the foliation $\mathcal{F}_2$ must have the form uv, where u and v are the coordinates along the axes defined by the basis

$$\left\{ \left(\frac{-\sqrt{2}-2}{2}, 1 \right), \left(\frac{\sqrt{2}-2}{2}, 1 \right) \right\},$$

which diagonalizes the jacobian

$$J_{\mathcal{F}_2}(0,0) = \begin{bmatrix} -32 & -16 \\ 32 & 32 \end{bmatrix}.$$

Hence,

$$u = \frac{y\sqrt{2} - 2y - 2x}{2\sqrt{2}} \quad \text{and} \quad v = \frac{y\sqrt{2} + 2y + 2x}{2\sqrt{2}},$$

so that

$$uv = \frac{-y^2 - 4xy - 2x^2}{4};$$

from which we deduce that the corresponding algebraic solution must be of the form

$$-y^2 - 4xy - 2x^2 + a_2x^3 + a_3x^2y + a_4xy^2 + a_4y^3.$$

Applying to this polynomial the usual algorithm, based on undetermined coefficients, we find that the desired algebraic solution is defined by the polynomial

$$2y^3 + (6x+1)\, y^2 + \left(6x^2 + 4x\right)\, y + 2x^3 + 2x^2.$$

5.3. **Example.** Consider now the foliation $\mathcal{F}_3$, for which

$$\begin{aligned} a_3 &= 36y^2x - 36y^2 + 36yx^2 - 18yx - 162y + 18x^3 - 108x \\ b_3 &= 36y^3 + 36y^2x + 144y^2 + 18yx^2 + 162yx + 216y + 36x^2 + 162x, \end{aligned}$$

This time, the Gröbner factorization algorithm returns 14 systems; of these 11 directly give a value for m. Ignoring the fractional values, we get $m = 1$, 2, 3 or 4. However, there are also two systems with no pure equations on m, one of which is

$$2m - \nu_3 - \mu_3 - 2 = 3{\nu_3}^2 + (6\mu_3 - 8)\nu_3 + 3{\mu_3}^2 - 8\mu_3 + 16 = 0. \tag{5.1}$$

Diagonalizing the degree two polynomial and re-writing the degree one polynomial in the same coordinates, we end up with the system

$$m - \frac{y_2}{\sqrt{2}} = \frac{\left(6{y_2}^2 + 16\right)\sqrt{2} - 16y_2}{\sqrt{2}} = 0.$$

Since the quadratic equation does not have any real solutions, this system does not contribute any new degrees to the ones we have already obtained above, and the same holds for the other system. Finally, using the usual strategy of undetermined coefficients, one easily shows that the only solution of degree one is $y+x$, while the only solutions of degrees 2 and 3 are powers of $y+x$, hence reducible. This leaves only the case of degree 4, for which the usual algorithm is too slow. So we use a strategy similar to the one already employed when we analysed $\mathcal{F}_2$.

Looking at the systems obtained when the Gröbner factorization algorithm is applied to the equations returned by `indexSums`, we discover that the only one which contains $m-4$ also contains $\mu+\nu=2$, where $[\mu,\nu]$ is the LL-pair at the origin. Since the jacobian J of $\mathcal{F}_3$ at $(0,0)$ is not diagonalizable, we determine the matrix

$$M = \frac{1}{162}\begin{bmatrix} -162 & 162 \\ 162 & -161 \end{bmatrix} \quad \text{such that} \quad M^{-1}JM = \begin{bmatrix} 54 & 1 \\ 0 & 54 \end{bmatrix}$$

is a Jordan block. The coordinates of $X=(x,y)^t$ in the basis whose vectors are the columns of M are $MX=(162y+161x, 162y+161x)^t$. Therefore, any algebraic solution of degree 4 of $\mathcal{F}_3$ must have multiplicity 2 at the origin, and homogeneous component of order 2 equal to $(162y+161x)^2$. Using undetermined coefficients we find that there is no algebraic solution satisfying these constraints. Therefore, the only polynomial, irreducible over $\mathbb{Q}[x,y]$, that is invariant under $\mathcal{F}_3$ is $y+x$.

5.4. **Example.** As a fourth example, let $\mathcal{F}_4$ be the foliation defined by

$$\begin{aligned} a_4 &= yx^2 - 2x + 1 \\ b_4 &= -y^2 + 9yx - 7y + x^3 - 7x^2 + 3x + 2. \end{aligned}$$

The singularities of this foliation can be broken into a block of degree 6 and a block which consists of the dicritical singularity $(1,1)$. Applying the Gröbner factorization algorithm to the equations returned by `indexSums` we get the systems

$$[[m,\ \nu+\mu],\ [m-4,\ \nu+\mu-2],\ [m-1,\ \nu+\mu+1],\ [m-3,\ \nu+\mu-3.]]\,,$$

where $[\mu,\nu]$ is the LL-pair at $(1,1)$. In particular, any algebraic solution this foliation might have cannot be of degree larger than 4. But, as in the previous examples, these systems tell us more than just that. For example, an algebraic solution of degree 3 for $\mathcal{F}_4$ must have multiplicity 3 at $(1,1)$. Hence, in terms of coordinates centered at $(1,1)$, this must be a homogeneous polynomial. Searching for such a polynomial we find

$$f = (y-1)^3 - 9(x-1)(y-1)^2 + 6(x-1)^2(y-1) + (x-1)^3.$$

However, since f is homogeneous with respect to the coordinates $u=x-1$ e $v=y-1$, it can be factored as a product of three linear polynomials. In other words, the corresponding algebraic solution can be decomposed as the union of three lines. At first sight this may seem surprising, because the only system for which $m=1$ contains the equation $\nu+\mu+1=0$, which does not have non-negative integer solutions. However, these systems apply only to curves defined and irreducible over the effective base field which, in these examples, is $\mathbb{Q}$. Indeed, computing the algebraic solutions of degree one of $\mathcal{F}_4$, we get

$$(y-1)+\alpha(x-1) \quad \text{where} \quad \alpha^3+9\alpha^2+6\alpha-1=0. \tag{5.2}$$

It is easy to show, using AXIOM , that f is indeed the product of the three linear polynomials defined in (5.2). The only other possibility is a curve of degree 4 and multiplicity 2 at the point $(1,1)$. But a search for algebraic solutions of degree 4 returns only homogeneous polynomials, showing that such a curve does not exist.

5.5. **Example.** Our next example is the foliation $\mathcal{F}_5$, defined by

$$a_5 = (4x-4)y^3 + \big(6x^2 - 12x + 6\big)y^2 + \big(6x^3 - 12x^2 + 6x\big)y + 2x^4 - 8x^3 - 2$$
$$b_5 = 4y^4 + (6x-6)y^3 + \big(6x^2 - 6x\big)y^2 + \big(2x^3 - 6x^2\big)y - 2x^3 + 2.$$

The singularities of $\mathcal{F}_5$ split into five blocks, three of them with 6, 3 and 2 singularities each, and two blocks of one singularity. The last two correspond to the dicritical singularities $(0,1)$ and $(-1,1)$; all the other singularities being non-dicritical. Applying `indexSums` to $\mathcal{F}_5$ we get 32 pairs of equations. Since none of these pairs contain an equation that is pure in m, we will have to diagonalize the quadratic equations obtained by the application of `groebnerFactorize`, as explained in section 4. For example, diagonalizing the homogeneous component of degree two of

$$58989\nu_5^2 - (22472\nu_4 - 117978\mu_5 + 22472\mu_4 + 88828)\nu_5 + 58989\nu_4^2 -$$
$$(22472\mu_5 - 117978\mu_4 + 88828)\nu_4 + 58989\mu_5^2 - (22472\mu_4 + 88828)\mu_5 + 58989\mu_4^2$$
$$- 88828\mu_4 + 239164 = 0.$$

and writing the corresponding linear equation in the same coordinates we get

$$m - \frac{4y_3}{5} = 140450y_4^2 + 95506y_3^2 - 177656y_3 + 239164 = 0.$$

Completing squares, we find that $y_3 \leq 2$. Thus,

$$m = \frac{4y_3}{5} \leq \frac{8}{5};$$

so that $m \leq 1$. Proceeding in the same way with the other 31 pairs of equations, we find that $m \leq 5$. Moreover, when we solve the linear diophantine equations for the admissible values of m, we find that they all have a finite number of solutions. However, since $\mu_i + \nu_i$ represents the order of a curve at one of its points, it cannot exceed the degree m of the curve. But it turns out that none of the solutions of the linear diophantine equations satisfies both the corresponding quadratic equations and the inequalities $\mu_i + \nu_i \leq m$, when $i = 5, 4$. Therefore, $\mathcal{F}_5$ is a foliation of $\mathbb{P}^2$, with dicritical singularities, that does not have any algebraic solutions.

5.6. **Example.** As a final example consider the degree two foliation $\mathcal{F}_6$ defined by the 1-form

$$\omega_6 = (y^3 - 2y^2 + yx^2 - y - x^2 + 2)dx - (y^2x - 2yx + x^3 - 3x^2 + 3x)dy.$$

This time there are seven blocks of degree one, corresponding to the singularities

$$\left(\frac{6}{5}, \frac{7}{5}\right), \left(\frac{3}{2}, \frac{1}{2}\right), (2,1), (0,1), (0,2), (1,1), \text{ and } \quad (0,-1),$$

the last three of which are dicritical. Applying `indexSums` to these blocks we get ten pairs of equations in the μs and νs, none of which gives a direct value for m. The diagonalized degree two polynomials are all of them of the form

$$8y_8^2 + 8y_7^2 + 8y_6^2 - c = 0$$

where $c \in \{0, 2, 4, 12\}$. However, the polynomial that defines m, with respect to the diagonalized coordinates, depends on all the six variables $y_1, y_2, y_3, y_4, y_5, y_6$. Thus, no bound on m can be obtained for $\mathcal{F}_5$ using the strategy of section 4. Indeed, all the ten pairs of diophantine equations on m and the μs and νs, have many solutions for small integer values of the variables, suggesting that $\mathcal{F}_6$ may have infinitely many algebraic solutions. Searching for algebraic solutions of degree two and order one at $(0, -1)$, one finds, among others, the following four polynomials

$$\begin{aligned}
f_1 &= -\frac{1}{3}y^2 + \left(\frac{1}{6}x + \frac{1}{3}\right)y + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{2}{3},\\
f_2 &= -\frac{1}{3}y^2 + \left(\frac{1}{9}x + \frac{1}{3}\right)y + \frac{1}{3}x^2 - \frac{10}{9}x + \frac{2}{3},\\
f_3 &= -\frac{1}{2}y^2 + xy - x + \frac{1}{2},\\
f_4 &= -\frac{1}{2}y^2 + \frac{1}{2}xy - \frac{1}{2}x + \frac{1}{2},
\end{aligned}$$

whose co-factors are, respectively,

$$\begin{aligned}
g_1 &= 2y^2 - 3y + 2x^2 - 2x + 1, & g_2 &= 2y^2 - 3y + 2x^2 - 3x + 1,\\
g_3 &= 2y^2 - 4y + 2x^2 - 2x & g_4 &= 2y^2 - 4y + 2x^2 - x.
\end{aligned}$$

Since $g_1 - g_2 + g_3 - g_4 = 0$, it follows from Darboux's method [12, section 25G, p. 493] that

$$\frac{f_1 \cdot f_3}{f_2 \cdot f_4} = \frac{3}{2}\frac{2y^2 - (3x+2)y - 2x^2 + 9x - 4}{3y^2 - (x+3)y - 3x^2 + 10x - 6}$$

is a rational first integral of $\mathcal{F}_6$, confirming that it has infinitely many algebraic solutions, as expected.

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