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The American Mathematical Monthly, Vol. 110, No. 4. (Apr., 2003), pp. 298-313.

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S. C. Coutinho and J. C. McConnell

1. INTRODUCTION. Articles on the history of mathematics can be written from many different perspectives. Some aim to survey a more or less wide landscape, and require the observer to watch from afar as theories develop and movements are born or become obsolete. At the other extreme, there are those that try to shed light on the history of particular theorems and on the people who created them. This article belongs to this second category. It is an attempt to explain Goldie's theorems on quotient rings in the context of the life and times of the man who discovered them.

2. FRACTIONS. Fractions are at least as old as civilisation. The Egyptian scribes of 3,000 years ago were very skilful in their manipulation, as attested by many ancient papyri. To the Egyptians and Mesopotamians, fractions were just tools to find the correct answer to practical problems in land surveying and accounting.

However, the situation changed dramatically in Ancient Greece. To the Greek philosophers, number meant positive integer, and 1 was 'the unity', and as such, had to be indivisible. So how could 'half' be a number, since 'half the unity' did not make sense? Possibly as a consequence of that, the Greek mathematicians thought of fractions in terms of ratios of integers, rather than numbers.

After the demise of Greek civilisation, mathematicians reverted to the more prosaic view that fractions were numbers. Indeed, for the next thousand years everyone seemed happy to compute with all sorts of 'numbers' without worrying much about what a number was really supposed to be.

It was the need for a sound foundation for the infinitesimal calculus that put mathematicians face to face with the nature of numbers. The movement began in the eighteenth century, but its first fruits were only reaped in the nineteenth century in the movement that became known as the *arithmetization of analysis*. In short, mathematicians felt quite sure that they knew their integers very well; so they thought that by constructing the real numbers in terms of positive integers they would place the latter on a firm foundation. The most extreme version of this credo is illustrated by Krönecker's well-known remark: "God made the integers, all the rest is the creation of man."

The construction of the reals from the integers proceeds in several stages: first axiomatize the positive integers, then construct negative from positive integers, then rationals from integers, and finally reals from rationals. In order to construct the rationals from the integers the old Greek idea of ratio was revived and cloaked in modern garb. A text on the theory of real functions published in 1886, and quoted in [**27**, p. 104], has this to say about fractions:

> A fraction cannot be regarded anymore as the union of equal parts of the unity; the expression *parts of the unity* does not make sense; a fraction is a set of two numbers, arranged in a certain order; for this new type of number it is necessary to recover the definitions of equality, inequality and the arithmetic operations.

A tentative representation of rationals as pairs of integers was given by Weierstrass in his lectures in the 1860s. Peano independently followed a similar approach. In a paper of 1891 he explains his theory in detail, but to us his definition of the equality of two fractions seems incredibly stilted. By then the definition that we are most used to had already been given by O. Stoltz in his book *Vorlensugen über Arithmetik.* Amusingly Peano comments in [**27**, p. 105] that Stoltz's definition seems to him "less simple" than his own.

Thus by the end of the nineteenth century, a fraction was thought of as a pair of integers, usually written in the form p/q, with q nonzero. Moreover, according to Stoltz, two fractions p/q and p'/q' were said to be equal if $pq' = p'q$, and their sum and product were defined by the formulas

$$p/q + p'/q' = (pq' + p'q)/qq', \quad p/q \cdot p'/q' = pp'/qq'.$$

About the same time, Kümmer, Dedekind, Krönecker, and others were helping create the theory of algebraic numbers, which in the next century would give rise to the theory of rings. Since much of the impetus for the development of abstract algebra in the early twentieth century came from Emmy Noether, it is not surprising to find fractions first being generalised to commutative rings by H. Grell, a member of her school. In the paper [**14**] of 1926, Grell defined what we now call the *quotient ring* (or *ring of fractions*) of a commutative domain. This is a ring whose elements are fractions (in the sense of Stoltz's definition) with numerator and denominator in the given domain.

Quotient rings of noncommutative rings were first considered in response to a question posed in section 12 of the first edition of van der Waerden's famous book *Moderne Algebra.* He asked whether a noncommutative domain can always be embedded in a division ring. The answer is no. A counterexample was provided by A. I. Malcev and is mentioned in the 1937 edition of van der Waerden's book.

Following a different approach, O. Ore had determined in 1931 a necessary and sufficient condition under which a noncommutative domain R could be embedded in a division ring Q, *under the additional hypothesis that Q is the quotient ring of R* [**25**, p. 466]. Since quotient rings play a fundamental role in the story we are going to tell, we will give a formal definition.

Let R be a ring with 1. We will not assume that the ring is commutative, nor that it is a domain. Recall that a nonzero element a of R is a *right zero-divisor* if there exists a nonzero a' in R such that $aa' = 0$; left zero-divisors are defined similarly. An element of R that is neither a right, nor a left, zero divisor is said to be *regular.* A ring Q is a *right quotient ring* of R if

(1) $R \subset Q$;
(2) every regular element c of R has an inverse c^{-1} in Q;
(3) every element of Q can be written in the form ac^{-1}, where a and c belong to R and c is a regular element.

Of course, if R is a domain then all of its nonzero elements are regular.

It is now easy to derive the necessary and sufficient condition that was found by Ore for the existence of a quotient ring. Suppose that Q is a right quotient ring of R, and that a and c are members of R. If c is regular then it is invertible in Q, so that $c^{-1}a$ belongs to Q. It then follows from our hypothesis on Q that there must exist a_1 and c_1 in R such that $c^{-1}a = a_1c_1^{-1}$, which gives $ca_1 = ac_1$. This suggests the following definition. A ring R satisfies the *right Ore condition* if, given a in R and c a regular element of R, there exist a_1 and c_1 in R such that $ca_1 = ac_1$ and c_1 is regular. It turns out that this condition is also sufficient.

Theorem 2.1. *A ring R has a right quotient ring if and only if R satisfies the right Ore condition.*

In Ore's 1931 paper [**25**] this theorem is proved only for domains. However, it caught the eye of P. Dubreil, who was at the time interested in the problem of embedding semigroups into groups, and in volume 1 of his 1946 book *Algèbre*, Dubreil proved the theorem in the form stated [**8**, Thèoreme 3, p. 147]. Another proof was given soon after by K. Asano [**3**].

If R is a domain, Ore's condition is easily seen to be equivalent to the simpler statement that if a and c are nonzero elements of R then $aR \cap cR \neq 0$. In this case every nonzero element of R becomes invertible in its quotient ring Q. Hence, if R is a domain, then Q is a division ring.

In another paper [**26**], published two years later, Ore returned to the same theme. This time he gave examples of noncommutative domains that satisfy the Ore condition. These are the rings that we now call *Ore extensions*. A very important particular case of the construction is the following. Let R be a domain and let d be a derivation of R. In other words, d is a homomorphism of the additive group of R and $d(aa') = ad(a') + d(a)a'$ for all a and a' in R. Ore showed that one can construct a 'twisted' polynomial ring $R[y; d]$, in one variable y, as follows. The elements of $R[y; d]$ are finite linear combinations of powers of y with coefficients in R. Thus, in order to define a multiplication in $R[y; d]$, it is enough to determine the relations satisfied by the product of an element of R by the variable y. In fact only one relation is postulated, namely, $ry = yr + d(r)$. Ore proved in [**26**, sec. 3] that if R is a division ring, then $R[y; d]$ satisfies the right Ore condition. A proof of the following more general result can be found in [**5**, Proposition 4, sec. 12.2].

Theorem 2.2. *If a domain R satisfies the right Ore condition, then so does $R[y; d]$.*

A simple but very important example of this construction is obtained by taking R to be the polynomial ring $k[x]$ over a field k of characteristic zero, and d to be differentiation with respect to x. The resulting Ore extension is called the *Weyl algebra*. This ubiquitous ring made its debut in a series of papers in the 1920s in which Dirac laid the foundations of his version of quantum mechanics. For more details see [**6**] and [**4**]. It is interesting to note that, in a 1931 paper dedicated to the ring theoretic properties of the Weyl algebra, D. Littlewood arrived at the Ore condition independently of Ore. The time must have been ripe for this work because it has been claimed in [**30**] that the condition was also known to the members of Noether's circle in Germany.

On the other hand, it is not difficult to see that there are noncommutative domains that do not satisfy Ore's condition. The easiest example is the free algebra $k\langle x, y\rangle$ over a field k. The elements of the right ideal $xk\langle x, y\rangle$ are linear combinations of monomials that begin with x, whereas those of $yk\langle x, y\rangle$ are linear combinations of monomials that begin with y. Therefore,

$$xk\langle x, y\rangle \cap yk\langle x, y\rangle = 0.$$

Ore's papers on ring theory were all published before 1945. After that he turned to graph theory. There were very few advances in the theory of quotient rings before Goldie began his work in this area in 1954. Indeed, when the two men later met, Ore said that he thought Goldie's work had rescued his own from oblivion.

Goldie's first breakthrough in the theory of quotient rings was the following theorem, which we prove in section 4.

Theorem 2.3. *A right Noetherian domain has a right quotient ring.*

3. ALFRED GOLDIE. Alfred William Goldie was born on 10 December 1920 in Coseley, Staffordshire. His father worked as a skilled fitter at Austin Motors, the car manufacturers. The factory employed 1,500 skilled workers serving 15,000 unskilled laborers. The former were responsible for preparing the brass templates used for the accurate positioning of the holes for the screws that held together the various parts of the car. Templates were prepared for new models every two years, and had to be accurate to one thousandth of an inch. The skilled workers often had little work to do, as the actual drilling was done by the unskilled employees.

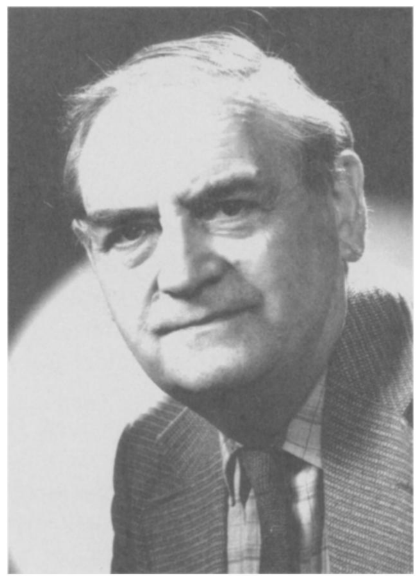

Figure 1. Alfred Goldie at the time of his retirement from Leeds.

Goldie attended Wolverhampton Grammar School, which he left in 1939 to read mathematics in Cambridge. He had a scholarship at St. John's College and, as a consequence, was exempted from the first year of the course. That explains why he graduated after only two years at Cambridge, instead of the usual three. By then Britain was fully engaged in the Second World War. In order to contribute to the war effort, Goldie was interviewed by C. P. Snow, the novelist known for his "Strangers and Brothers" sequence of novels and for his 1959 Rede Lecture "The Two Cultures," who was then working as scientific adviser to the Bristish Government. Goldie had been in the Officers Training Corps at Cambridge, and his training was regarded as equivalent to the standard six-week Certificate A infantry training. Snow, however, decided that Goldie could make a bigger contribution to the war effort by working in an intellectual field, rather than serving in the infantry. Thus he was sent to work in Cambridge with Dr C. A. Clemmow, the Superintendent of Ballistic Research. Soon after Goldie joined it the whole research team was evacuated to Shrewsbury.

Towards the end of the war, Goldie was transferred to London, where he worked on ballistics at the Woolwich Arsenal. About this time he met G. E. H. Reuter, who was at the time a research student working in Cambridge under Hardy and Littlewood. They discussed the possibility of Goldie's going back to Cambridge to finish his education by doing Part III of the Mathematical Tripos. However, he had been married in October 1944, and Reuter warned him that as a married man Goldie would not be able to afford another year as a student.

Since he was living in London, Goldie was allowed to start an internal London University Ph.D., and his employers at Woolwich helped by letting him have time off for study. At that time he had also begun attending the London Mathematical Society meetings. There he learned that Philip Hall had returned to Cambridge after working in Bletchley Park, and he was advised to write to Hall for possible research problems. An expert in group theory, Hall was probably the outstanding British algebraist of his time. Goldie wrote to Hall, and there followed a correspondence of several letters, and also visits to Hall's mother's home in Hampstead where Hall spent most weekends. Hall advised Goldie to read van der Waerden's *Moderne Algebra*, then only available in German; this Goldie did, with the help of a German dictionary!

In September 1946 Goldie left the Woolwich Arsenal to take a job as Assistant Lecturer at Nottingham University. This was a "tenure track" position, but the university had to decide whether one would be retained as lecturer with tenure, or sacked, after at most three years. At the time of the move to Nottingham, his wife Mary was pregnant, and their first child was born in October 1946.

Now that Goldie was no longer living in London, he had to discontinue his internal London University Ph.D. programme. Since Ph.D.'s were sought mostly because they helped one in getting an academic job, he never again tried to earn the degree. From Nottingham, Goldie wrote to Hall, who this time referred him to the first volumes of Bourbaki's *Algebra* that had just been published. Thus it came to pass that Goldie's first two papers dealt with the Jordan-Hölder theorem for general algebraic structures.

Since he was only an Assistant Lecturer in Nottingham, Goldie had a teaching load of up to twelve hours a week. Moreover, the conditions under which the Goldies were living were rather difficult. Fortunately, he received a new job offer, this time from Newcastle, where W. Rogosinski and the applied mathematician A. Green had just been appointed to professorships. As part of their strategy to build up the mathematics department, Rogosinski and Green appointed the algebraist K. A. Hirsch to a Readership, and both F. F. Bonsall and Alfred Goldie to Lecturerships. Newcastle had another attraction beside the offer of a permanent job: there Goldie would also be able to get a better house at an affordable rent.

Having exhausted the area of general algebraic structures on which he had been working, Goldie was advised by Hirsch to "do something else." There was a lot of group theory being done in Britain at the time, so he turned his attention to ring theory instead. The presence of Bonsall and Hans Reiter in Newcastle led him to work for a while on the algebras of analysis. He learned about normed rings from Reiter, and collaborated with Bonsall on two papers.

However, Goldie felt hampered in his work on analysis by his lack of feel for the applications. Moreover, Bonsall had been promoted to a Readership in the meantime, and Goldie thought that it would be better to return to pure algebra, where he stood a better chance of doing more independent work.

One of the most characteristic features of ring theory in the late 1940s and early 1950s was the search for a structure theory for general rings. The most successful approach to this problem was that of N. Jacobson. Thus it is not really surprising that Goldie's early work in ring theory was influenced by Jacobson's papers and books.

The basic concepts of Jacobson's theory are easily formulated. Given a ring R and a right ideal $\mathfrak{m}$ of R, set $(\mathfrak{m} : R) = \{x : Rx \subset \mathfrak{m}\}$. A two-sided ideal I of R is *right primitive* if it is contained in a maximal right ideal $\mathfrak{m}$ such that $(\mathfrak{m} : R) = I$. If 0 is a right primitive ideal, we say that R is a *right primitive ring*. A more transparent way to define these rings is available if we use the terminology of representation theory. Thus R is a right primitive ring if and only if it has a faithful irreducible representation.

Since $(\mathfrak{m} : R)$ is always a two-sided ideal, it follows that a ring whose only proper two-sided ideal is 0 must be primitive. Rings with this property are called *simple*; examples include matrix rings over fields and the Weyl algebra defined in section 2.

A very important two-sided ideal of a ring R is its *Jacobson radical* $J(R)$, which is the intersection of all the right primitive ideals of R. In [**16**] a ring is called *semisimple* if its Jacobson radical is zero. Since the Jacobson radical of the factor ring $R/J(R)$ is always zero, we have a canonical way of constructing a semisimple ring out of any ring R. Jacobson showed in [**18**, Theorem 1, p. 14] that every semisimple ring S can be embedded in a direct product of primitive rings in such a way that the projection of S over any direct factor is surjective. Another result that emerged from Jacobson's work in the 1940s was his proof in [**17**] that the set of primitive ideals of a ring R can be given a topology, similar to the well-known Zariski topology of commutative algebra. Jacobson called this topological space the *structure space* of R.

Goldie's first paper [**9**] on his return to pure algebra was a study of the properties of semisimple rings. The second theorem in the paper relates the structure space of a ring to that of a two-sided ideal (considered as a ring without unit). This result appears as Proposition 3 on page 206 of the first edition of Jacobson's book *Structure of Rings* [**18**] with a footnote explaining that it has been communicated by Goldie.

4. NOETHERIAN DOMAINS. Interest in quotient rings had briefly flared again in the early 1950s. In 1953, Dov Tamari [**31**] proved that the enveloping algebra of a finite dimensional Lie algebra satisfies the right Ore condition, and S. A. Amitsur showed in [**1**, Lemma 1, p. 465] that the same holds for PI-domains. Amitsur's result did not come to Goldie's attention at the time. Tamari's result, on the other hand, can be viewed as a direct ancestor of Goldie's theorem, so we discuss it in some detail.

The enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ is defined in terms of generators and relations: the generators are a basis of the Lie algebra $\mathfrak{g}$, and the relations come from the Lie product. In 1937, G. Birkhoff and E. Witt had independently shown how one can construct a vector space basis of $\mathcal{U}(\mathfrak{g})$ from its generators and relations. Tamari's proof follows the same approach, and depends on one's being able to find solutions to certain equations in $\mathcal{U}(\mathfrak{g})$.

Since enveloping algebras play a major role in the representation theory of Lie algebras, their ring theoretic properties were well known. For example, it follows from the work of Birkhoff and Witt that enveloping algebras are domains, and also that they satisfy a very weak 'finite dimensionality' condition: they are Noetherian rings. These rings are named after Emmy Noether, from whose work they sprung in the first half of the twentieth century. A ring R is *right Noetherian* if its right ideals are finitely generated. It is not difficult to prove that this is equivalent to either of the following conditions:

- R has no infinite strictly ascending chain of right ideals;
- every nonempty set of right ideals of R admits a maximal element with respect to inclusion.

At the International Congress of Mathematicians held in Amsterdam in 1954, Goldie had met Jacobson and had been encouraged to try to prove a conjecture of Jacobson's concerning Noetherian rings. The conjecture, which appeared in [**18**, p. 200], stated that, if R is a right and left Noetherian ring, then the intersection of all the powers of its Jacobson radical is zero; that is,

$$\bigcap_{n=1}^{\infty} J(R)^n = 0.$$

This had been proved for commutative rings by W. Krull. In 1965 I. N. Herstein [**15**] showed that it is false if the ring is right but not left Noetherian. The original question is still open, with the best positive result to date obtained by A. V. Jategaonkar in [**20**].

The conjecture required the study of primitive Noetherian rings, and Goldie decided to tackle first the case of simple Noetherian domains. He was acquainted with the work of Ore on quotient rings, as well as with Littlewood's paper on the Weyl algebra and Tamari's on enveloping algebras. So it seemed natural to try and use the Ore condition to show that a right Noetherian domain (whether simple or not) always has a right quotient ring. This is Theorem 2.1, which we stated at the end of section 2. Goldie proved it in 1956, and felt that it must be a major advance, a view shared by Bonsall.

It is now easy to give a short proof of Goldie's result on Noetherian domains. Recall that, if R is a domain, then the Ore condition is equivalent to the fact that $aR \cap cR \neq 0$ whenever a and c are nonzero elements of R. Now assume, by contradiction, that $aR \cap cR = 0$. We claim that, in this case, the sum

$$aR + caR + c^2aR + \cdots + c^naR$$

is direct for every $n \geq 1$. Indeed, if the sum is not direct, then there exist $r_0, \ldots, r_n$ in R, not all zero, such that

$$ar_0 + car_1 + c^2ar_2 + \cdots + c^nar_n = 0.$$

Thus

$$car_1 + c^2ar_2 + \cdots + c^nar_n = -ar_0.$$

But we are assuming that $aR \cap cR = 0$, so $r_0 = 0$ and

$$car_1 + c^2ar_2 + \cdots + c^nar_n = 0.$$

Since $c \neq 0$ and R is a domain, we can cancel c. Accordingly,

$$ar_1 + car_2 + \cdots + c^{n-1}ar_n = 0,$$

and the claim follows by induction. The existence of these direct sums implies that each inclusion in the ascending chain

$$aR \subset aR \oplus caR \subset aR \oplus caR \oplus c^2aR \subset \cdots$$

is strict. In particular, R cannot be a Noetherian ring. Independent confirmation that Goldie was correct in his evaluation of the importance of this result would come soon.

When, in 1952, Richard Brauer accepted a chair at Harvard University, he was already one of the leading figures on the international mathematical scene. He had been responsible for major advances in the theory of finite dimensional algebras and was the discoverer of modular representation theory.

Brauer's first academic post in Germany had been at the University of Königsberg, from which he was eventually dismissed by the Nazis in 1933. One of his colleagues there had been W. Rogosinski, and the two had become good friends. After the war, whenever Brauer visited England he would go to Newcastle and stay with the Rogosinskis. On one of these occasions, Goldie showed Brauer his result on Noetherian domains. All Brauer said was: "Oh my God!" Goldie then explained that he now needed to "do the matrix case"; Brauer replied "my boy, you are on your own."

To put Brauer's reaction in perspective one should bear in mind that although commutative Noetherian rings had been extensively studied—the first volume of Zariski and Samuel's *Commutative Algebra* would appear only two years later—the property had played a comparatively minor role in noncommutative ring theory. Add to that the fact that all known proofs of the Ore condition up to that point had required a detailed knowledge of the structure of the ring in question and it becomes easier to understand why Goldie's result seemed so unexpected to Brauer.

Let Goldie himself tell the the story to its conclusion:

> So I spent months representing faithfully primitive rings with 1 (even domains) and eventually came up with [the fact that] essential right ideals have regular elements—the rest was easy. The rationalisation of preparing the paper led at once to prime rings, then to "latitude and longitude" replacing Noetherian, thus ascending chain conditions on direct sums and on annihilators.

In the next two sections we flesh out the main elements in Goldie's story, and then show how they are put together to prove that a prime Noetherian ring always has a quotient ring.

5. PRIME RINGS. In this section and the next we abandon the historical approach. As we have seen, Goldie was led to the theorems by what with hindsight looks like a more or less tortuous path since, in his own words, he was really looking for "a more restrictive theorem that would give stronger information about primitive (even simple) rings." Fortunately, for us, there are now very crisp proofs of the theorems that also make apparent their impressive inner logic. Henceforth, we will let the theorems themselves take centre stage and tell their own story.

When first facing the theorems in their most general form, one may easily be put off by their seemingly "unnatural" hypotheses. Rather than risk that, we will consider first the more natural case of prime right Noetherian rings. Recall that a ring R is *prime* if, whenever I and J are two right ideals of R such that $IJ = 0$, either $I = 0$ or $J = 0$.

Domains are clearly examples of prime rings, but not all (noncommutative) prime rings are domains. For example, the ring of $n \times n$ matrices over a field is a prime ring, but as every linear algebra student knows, this ring has zero-divisors. More generally, a matrix ring over a domain (commutative or not) is always a prime ring. Indeed, a nice toy model to have in mind is the ring $M_n(\mathbb{Z})$ of $n \times n$ matrices with integer entries. If A belongs to $M_n(\mathbb{Z})$ and A' is its matrix of cofactors, then $A \cdot A' = det(A)I$, where I is the $n \times n$ identity matrix. It follows from this that A is a regular element of $M_n(\mathbb{Z})$ if and only if A has nonzero determinant. Thus all matrices of $M_n(\mathbb{Z})$ with nonzero determinants must be invertible in its quotient ring. It is easy to see from this that the quotient ring of $M_n(\mathbb{Z})$ is in fact $M_n(\mathbb{Q})$.

Since a matrix ring over a domain is always a prime ring, it seems reasonable to ask whether the converse is also true. Although the answer to this question is no, Goldie proved what may be described as "the next best thing": a prime Noetherian ring always has a quotient ring that is a matrix ring over a division ring. By 1957 Goldie had most of the ingredients for a proof of this result. Moreover, he had managed to pinpoint the missing link, the piece that would turn out to be the crux of the whole argument. It is not difficult to determine what this might be by backtracking from the desired result, and that's what we are going to do.

For the remainder of this section we will assume that R is a prime right Noetherian ring. It follows from section 4 that, if R has a right quotient ring, then it must satisfy the right Ore condition. Thus if a and c are given elements of R, with c regular, there must exist a_1 and c_1 in R such that $ca_1 = ac_1$ and c_1 is regular. In particular, if $a \neq 0$ then $ca_1 = ac_1$ is a nonzero element of $cR \cap aR$. Thus our first hurdle consists in proving the following claim:

Claim 5.1. *If c is a regular element of a prime right Noetherian ring R and if a in R is nonzero, then $cR \cap aR \neq 0$.*

One's first reaction may be: this is just what we have proved for domains in section 4—could we not use a similar approach? The answer is yes. First, we introduce some terminology:

- A right ideal E of R is *essential* if $E \cap I \neq 0$ for every nonzero right ideal I of R.
- A right ideal U of R is *uniform* if it is nonzero and if the intersection of any two nonzero right ideals of R contained in U is also nonzero.

It follows from the argument given in section 4 that every nonzero right ideal of a Noetherian domain is uniform. In the prime Noetherian ring case the corresponding result is somewhat weaker.

Lemma 5.2. *Every nonzero right ideal of a prime right Noetherian ring R contains a uniform right ideal.*

Proof. Assume that the result is false, and that I is a nonzero right ideal of R that does not contain a uniform right ideal. Then I contains two nonzero right ideals I_1 and I_1' such that $I_1 \cap I_1' = 0$. Now I_1 does not contain a uniform ideal; otherwise I would contain such an ideal, thus contradicting the hypothesis. Hence there must be two nonzero right ideals I_2 and I_2' contained in I_1 such that $I_2 \cap I_2' = 0$. Moreover, since $I_1 \cap I_1' = 0$ and I_2' lies in I_1, it follows that the sum $I_1' + I_2'$ is direct.

Continuing in this way we can construct right ideals $I_1', I_2', I_3', I_4', \dots$ such that

$$I_1' \oplus I_2' \oplus I_2' \oplus I_3' \oplus \cdots$$

is an infinite direct sum of nonzero right ideals in R. But this contradicts the right Noetherian condition. ∎

Now let I be a right ideal of R, and let $\mathcal{S}$ be the set of direct sums of uniform right ideals contained in I. Lemma 5.2 implies that $\mathcal{S}$ is nonempty, and it follows from the Noetherian hypothesis that $\mathcal{S}$ must have a maximal element. Thus, I contains (finite) maximal direct sums of uniform right ideals, where "maximal" means that no other direct summand can be added to the sum. Now take $I = R$. Since a maximal direct sum of uniform right ideals of R must be essential, this gives a simple method for constructing essential right ideals of R.

In [**10**, Theorem 6, p. 598] Goldie showed that the number of summands in a maximal direct sum of uniform ideals contained in a right ideal I of R is an invariant of I. The argument is similar to that used to prove that two bases of a finite dimensional vector space always have the same number of elements, so we will omit it; the details can be found in [**24**, Theorem 2.2.9]. This invariant is now called the *uniform rank* or *Goldie rank* of I. Claim 5.1 is an immediate consequence of the next lemma.

Lemma 5.3. *If c is a regular element of a prime right Noetherian ring R, then cR is an essential right ideal of R.*

Proof. Let c be a regular element of R. Since R is Noetherian, it must have finite uniform rank, say n. Thus there exist uniform ideals $U_1, \ldots, U_n$ of R such that $U_1 \oplus \cdots \oplus U_n$ is a maximal direct sum in R. But c is a regular element of R and each U_i is nonzero, whence $cU_i \neq 0$ for $1 \leq i \leq n$.

Moreover, a nonzero right ideal contained in cU_i must be of the form cI, where I is a right ideal of R contained in U_i. If cJ is another nonzero right ideal contained in U_i, then $I \cap J \neq 0$ because U_i is uniform. Since c is a regular element, it follows that

$$cI \cap cJ = c(I \cap J) \neq 0.$$

Thus $cU_1, \ldots, cU_n$ are uniform right ideals of R.

Since R has uniform rank n, it follows that $cU_1 \oplus \cdots \oplus cU_n$ is a second maximal direct sum of uniform ideals of R. Now, suppose that L is a nonzero right ideal of R. By Lemma 5.2, L contains a uniform right ideal V. Because $V \neq 0$ and the sum $cU_1 \oplus \cdots \oplus cU_n$ is maximal, we can infer that

$$0 \neq (cU_1 \oplus \cdots \oplus cU_n) \cap V \subset cR \cap L,$$

which proves the lemma. ■

Unfortunately, $cR \cap aR \neq 0$ implies only that there exist nonzero elements b_1 and b_2 in R such that $cb_1 = ab_2$. However, in order to prove the Ore condition we must be able to choose b_1 and b_2 with b_2 ***regular in*** R. In other words, we want to demonstrate that

$$E = \{x \in R : ax \in cR\}$$

contains a regular element. But E is a right ideal of R. Moreover, since cR is essential (Lemma 5.3), it follows that E is likewise essential. The proof is easy: Suppose that I is a nonzero right ideal of R, then either $aI = 0$ or $aI \neq 0$. In the former case, I is a subset of E; in the latter, $aI \cap cR \neq 0$, which implies that $I \cap E \neq 0$. Thus what seems to be at stake here is some sort of converse of Lemma 5.3. Indeed, for our purposes, it would be enough to establish the following conjecture.

Conjecture 5.4. *Every essential right ideal of a prime right Noetherian ring R contains a regular element.*

At first it might appear that we are trying to jump too far, but one quickly sees that it is not so. Namely, we still have not made use of the fact that we expect the quotient ring Q of R to be a matrix ring over a division ring. Let us assume that this is the case, and also that E is a nonzero essential right ideal of R. Now, if I is a nonzero right ideal of Q, then by clearing denominators we conclude that $I \cap R \neq 0$. Since E is essential in R, it follows that $I \cap R \cap E \neq 0$. Therefore, EQ is essential in Q. But a well-known property of matrix rings over division rings asserts that, if J is any proper right ideal of Q, then $J \oplus J' \cong Q$ for some right ideal J' of Q. This implies that if EQ is essential, then it is actually equal to Q. In particular, $1 = ac^{-1}$ for some a in E, and clearing denominators we conclude that $c = a$ belongs to E. Here's what we have proved.

Proposition 5.5. *If a prime right Noetherian ring R has a quotient ring that is a matrix ring over a division ring, then every essential right ideal of R contains a regular element.*

6. THE ESSENTIAL RESULT. Sometime after Goldie had realised the importance of Conjecture 5.4, he took his family on a week's camping trip to Sedbergh. This historic town is situated in the Yorkshire Dales National Park, close to the rugged fells of the Lake District. But things did not go quite as planned, for it rained often and there was nowhere to dry clothes or the baby's nappies (diapers). After five days they decided to give up, and returned to Newcastle on a Thursday. The break must have given Goldie's subconscious time to catch up, for between Friday and Monday he proved Conjecture 5.4.

As in section 5 we will not follow Goldie's approach in [**10**]. Instead, we will present a simpler proof that Goldie gave several years later (see [**12**, sec. 1]). We begin with some definitions. Let R be a ring, and let a be an element of R. The *right annihilator of* a is the right ideal

$$r(a) = \{x \in R : ax = 0\}.$$

The *left annihilator* $\ell(a)$ is defined analogously. Note that a is regular if and only if $r(a) = \ell(a) = 0$. An easy property of right annihilators that we will use several times is the following: if a and x are members of R, then $r(a)$ is contained in $r(xa)$. Finally, recall that an element a of R is *nilpotent* if $a^n = 0$ for some positive integer n. We need two technical lemmas before we tackle the conjecture.

Lemma 6.1. *Let R be a prime right Noetherian ring, and let I be a right ideal of R. If all the elements of I are nilpotent, then $I = 0$.*

Proof. Suppose that there exists a nonzero element a in I. We argue to a contradiction. Since R is right Noetherian, the set of right ideals

$$\mathcal{S} = \{r(za) : z \in R \text{ and } za \neq 0\}$$

must have maximal elements. Choose z in R such that $b = za$ and $r(b)$ is maximal in $\mathcal{S}$. Let x belong to R. Since axz is in I, it follows that $(axz)^n = 0$ for some positive integer n. Thus

$$(xb)^{n+1} = (xza)^{n+1} = xz(axz)^n a = 0,$$

so xb is also nilpotent. Choose the smallest positive integer k such that $(xb)^k = 0$. Since

$$r(b) \subset r((xb)^{k-2}xb) = r((xb)^{k-1}),$$

it follows from the maximality of $r(b)$ that $r(b) = r((xb)^{k-1})$. However, $(xb)^k = 0$ implies that xb is in $r((xb)^{k-1})$, so that $bxb = 0$. Since this holds for all x in R, it follows that $bRb = 0$. Hence $(bR)^2 = 0$. Exploiting the fact that R is a prime ring, we conclude that $b = 0$, which is a contradiction. ∎

Lemma 6.2. *Let R be a prime right Noetherian ring. If a is in R, then $a^n R \oplus r(a^n)$ is an essential right ideal for all sufficiently large positive integers n.*

Proof. Since R is right Noetherian, it follows that there exists $N > 0$ such that $r(a^n) = r(a^{n+1})$ for all $n \geq N$. We will prove first that for such an n the sum $a^n R + r(a^n)$ is direct. Choose x in R such that $a^n x$ belongs to $r(a^n)$. Thus $a^{2n}x = 0$

and, by the maximality of $r(a^n)$,

$$x \in r(a^{2n}) = r(a^n).$$

This implies that $a^n x = 0$. Therefore, $a^n R \cap r(a^n) = 0$ for all $n \geq N$.

Now we must show that $a^n R \oplus r(a^n)$ is essential. If not, then there exists a nonzero right ideal I of R such that $I \cap (a^n R + r(a^n)) = 0$. Note that $a^{kn} I \neq 0$ for all $k \geq 0$, because I is not contained in the right annihilator of a^n. We claim that the sum

$$a^n I + a^{2n} I + \cdots + a^{kn} I$$

is direct for all $k > 0$. Suppose by induction that the claim holds for $k - 1$. If

$$x \in a^n I \cap (a^{2n} I + \cdots + a^{kn} I),$$

then $x = a^n y = a^{2n} z$, where y is in I and z in R. Therefore, $a^n(y - a^n z) = 0$, placing $y - a^n z$ in $r(a^n)$. Thus

$$y \in I \cap (a^n R + r(a^n)) = 0.$$

In particular, $x = a^n y = 0$. Since a right Noetherian ring cannot have infinite direct sums of unrestricted length, the proof is complete. ■

We are now in possession of all the tools required to prove Conjecture 5.4. We will state the result so as to include Lemma 5.3.

Theorem 6.3. *Let R be a prime right Noetherian ring. A right ideal of R is essential if and only if it contains a regular element.*

Proof. The condition is sufficient by Lemma 5.3; we need prove only that it is necessary.

Let E be a nonzero right ideal of R. Since $E \neq 0$, it follows from Lemma 6.1 that it contains a non-nilpotent element x. Moreover, by Lemma 6.2 there exists $n > 0$ such that $x^n R \cap r(x^n) = 0$. Let $a_1 = x^n$. Then either $r(a_1) \cap E = 0$ or $r(a_1) \cap E \neq 0$. In the latter case, we repeat the same argument with $r(a_1) \cap E$ in place of E. Having done that, we find a nonzero element a_2 in $r(a_1) \cap E$ such that $a_2 R \cap r(a_2) = 0$. The conditions on a_1 and a_2 imply that the sum

$$a_1 R + a_2 R + (r(a_1) \cap r(a_2) \cap E)$$

is direct. If $r(a_1) \cap r(a_2) \cap E \neq 0$, the process continues.

In the kth step of this process we obtain a direct sum

$$a_1 R \oplus \cdots \oplus a_k R \oplus (r(a_1) \cap \cdots \cap r(a_k) \cap E),$$

where $0 \neq a_k \in r(a_1) \cap \cdots \cap r(a_{k-1}) \cap E$. Since R is right Noetherian, the process must stop, say at step k. But this will happen only if

$$r(a_1) \cap \cdots \cap r(a_k) \cap E = 0.$$

However, if we assume that E is essential, then

$$r(a_1) \cap \cdots \cap r(a_k) = 0.$$

Let $c_1 = a_1 + \cdots + a_k$. Since the sum $a_1 R \oplus \cdots \oplus a_k R$ is direct, it follows that

$$r(c_1) = r(a_1) \cap \cdots \cap r(a_k) = 0.$$

By Lemma 6.2, $c_1^n R \oplus r(c_1^n)$ is an essential right ideal of R for some $n > 0$. Let $c = c_1^n$. Since $r(c_1) = 0$, it follows that $r(c) = r(c_1^n) = 0$. We show that $\ell(c) = 0$. Let z be an element of $\ell(c)$. The ideal $r(z)$ contains the essential ideal

$$cR = cR \oplus r(c),$$

so that $r(z)$ is itself an essential right ideal of R. Applying Lemma 6.2 again, this time to z, we find that $z^m R \cap r(z^m) = 0$ for some positive integer m. Because $r(z)$ is contained in $r(z^m)$ and $r(z)$ is essential, we deduce that $z^m R = 0$. Thus $z^m = 0$. This shows that all the elements of $\ell(c)$ are nilpotent. Hence, by Lemma 6.1, $\ell(c) = 0$. Thus c is a regular element contained in E. ■

7. GOLDIE RINGS. The results we have established so far give a complete proof that prime right Noetherian rings always have quotient rings. Goldie also showed that such a quotient ring is isomorphic to a matrix ring over a division ring, but this part of the proof requires more ring theory than we are assuming on the part of the reader, so it will be omitted. The details can be found in [**12**].

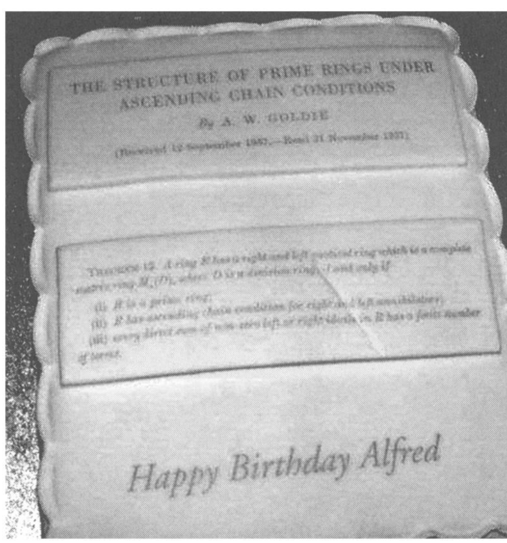

Figure 2. The cake for Goldie's 80th birthday celebration in Glasgow (April, 2001), with an excerpt from [**11**].

Having reached this point one may ask: Does the converse hold? If a prime ring R has a quotient ring, must R be Noetherian? The answer turns out to be no; but as Goldie showed, it is possible to give a complete characterization of those prime rings that have a quotient isomorphic to a matrix ring over a division ring. Indeed, if we scrutinize our proofs very carefully, we will see that the Noetherian property was only used as a means to ascertain two properties from which everything else follows. This is how the properties are stated in Goldie's 1958 paper "The Structure of Prime Rings under Ascending Chain Conditions" [**10**, p. 590]:

(1) Every direct sum of nonzero right ideals of R has a finite number of terms.

(2) The ascending chain condition holds for the annihilator right ideals of R.

In his M.Sc. course on ring theory, more than twenty years later, Goldie still referred to these as "rings that satisfy conditions (1) and (2)"; by then everyone else called them *right Goldie rings*. We may now state the theorem in full.

Theorem 7.1. *A ring R has a right quotient ring that is a matrix ring over a division ring if and only if R is a prime right Goldie ring.*

This is not the exact content of the theorem that appears in [**10**, Theorem 12, p. 606]. That's because Goldie assumed that R had both a right and a left quotient ring, so it had to satisfy not only (1) and (2), but also their left-handed counterparts. But Goldie was not the only mathematician working on Noetherian rings at the time, and building on Goldie's work, L. Lesieur and R. Croisot proved Theorem 7.1 in 1959 (see [**22**]). However, by then Goldie had moved even further.

It may appear that Theorem 7.1 does not admit of improvement—after all, it gives a necessary and sufficient condition for a ring to have a quotient ring of the appropriate type. However, we are assuming that the ring is prime, which raises the question: Would it be possible to find a similar result for nonprime rings? If we scan our proofs once more, this time looking for uses of the prime hypothesis, we will find that it is necessary only to rule out nilpotent ideals. Recall that a right ideal I in a ring R is *nilpotent* if $I^n = 0$ for some positive integer n. A *semi-prime* ring is one whose only nilpotent ideal is 0. In his paper "Semi-prime Rings with Maximal Condition" [**11**], Goldie generalized his previous result to semi-prime rings.

Theorem 7.2. *A ring R has a right quotient ring that is a finite direct sum of matrix rings over division rings if and only if R is a semi-prime right Goldie ring.*

As has already been indicated, the results discussed in sections 5 and 6 are enough to give a complete proof that a semi-prime right Goldie ring has a quotient ring. A complete characterization of this quotient ring and a short proof of the converse can be found in [**12**].

Although Goldie rings have surfaced in a more or less spontaneous way, it is fair to ask if there are any natural examples of rings that are Goldie but not Noetherian. This was answered in a paper of E. Posner [**28**] published in 1960. Posner showed that prime rings satisfying a polynomial identity (also known as *prime PI rings*) are right and left Goldie. He also found a very nice characterization of the quotient ring in this special case.

It is time to take another look at our toy model, for $M_n(\mathbb{Z})$ is also a PI ring. Indeed, it is not difficult to write down explicit identities for these rings, some of which can be found in [**29**]. Now, if A belongs to $M_n(\mathbb{Z})$, then its matrix of cofactors A' also has integer entries. Thus it follows from $A \cdot A' = \det(A)I$ that, in order to invert A, it is enough to invert $\det(A)I$. As multiples of the identity matrix, such matrices commute with all the elements of $M_n(\mathbb{Z})$. Posner showed that a similar result holds for all prime PI rings. Two definitions are required before we state Posner's result. The *centre* $Z(R)$ of a ring R is the subring comprising those elements z of R for which $zx = xz$ holds for all x in R. Let K be a field. An algebra R over K is *central simple* if it has no proper two-sided ideals and if $Z(R) = K$.

Theorem 7.3 (Posner). *Let R be a prime ring that satisfies a polynomial identity. Then R has a right and left quotient ring Q that is a central simple algebra over its centre $Z(Q)$. Moreover, $Q = Z(Q)R$.*

Posner's proof of this result in his 1960 paper was rather unclear, and a clearer proof was given by S. Amitsur [**2**] in 1967 using ultrafilters, a concept borrowed from logic.

By then Goldie's life had changed dramatically. He had spent the academic year 1960–61 in Yale at Jacobson's invitation. His lectures there gave rise to the *Yale Notes*, which served as a handbook for many of his first research students. It was during this visit that Goldie first met O. Ore and I. N. Herstein. The latter was to play an important part in bringing the theory of noncommutative Noetherian rings to the attention of a wider public through his books, his students, and his many coworkers at Chicago.

Back in England, Goldie was appointed in 1963 to a Research Chair in Leeds University, with the purpose of building up research in mathematics at that institution. In 1968 Goldie's research students in Leeds were having a hard time in their attempt to understand both Posner's work and Amitsur's proof of Posner's theorem. They asked Goldie for help, and he agreed to give a lecture on the subject when he returned from a trip to Paris, where he would be meeting with the local mathematicians. On his arrival in Paris, Goldie found the city in the grip of the 1968 student protests. There was a huge procession going down the Boulevard St. Michel, and having no other choice, Goldie joined the crowd, carrying his luggage, and dropped out of the parade only when he reached his hotel. Since everything was at a standstill in Paris, his hosts took him to the IHÉS, the Institut des Hautes Études Scientifiques at Bures sur Yvette. Being more or less isolated, since there was no public transport, Goldie seized on the opportunity to concentrate on Posner's theorem. As a result of this effort he came up with the new and more intelligible proof that appeared in [**13**].

The Goldie theorems have led to a rapid flowering of the area of noncommutative ring theory. This includes many surprising analogues to results in commutative algebra theory, such as the Serre splitting theorem on the existence of free summands and the theorem of Bass on the cancellation of free summands. More unexpected developments were the applications to infinite-dimensional representations of groups, Lie algebras, Lie groups, and rings of differential operators. A large part of these developments can now be found in monographs such as [**24**] and, for the semisimple Lie algebra theory, [**7**] and [**19**].

8. POSTSCRIPT. Although this brings our story to an end, it in no way exhausts Goldie's contribution to mathematics. He continued to do groundbreaking work in ring theory and has played a key role in establishing noncommutative Noetherian ring theory as a recognisable subject with its own methods and problems. This is a subject, moreover, that has not remained isolated from the rest of mathematics.

Thanks to MathSciNet we can perform a simple statistical experiment to gauge the impact of Goldie's work by searching the database for the boolean expression 'Goldie and ring.' The output contains more than 800 entries!

Goldie retired from his chair in 1986, an event that was marked by a major conference in Leeds. However, after retirement, he continued—and now in his 80th year still continues—to do research in algebra.

ACKNOWLEDGMENTS. First and foremost we thank Alfred Goldie for providing so much information on the early history of the theorems, and for patiently answering all our questions. This paper is dedicated to him. We also wish to thank J. C. Robson and Lance Small for their many suggestions and comments on previous drafts of the paper. Finally, we are grateful to Professor Goldie and Ken Brown for the pictures that illustrate the paper.

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