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KRULL DIMENSION OF MODULES AND INVOLUTIVE IDEALS

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ABSTRACT. In this paper we establish an upper bound for the Krull dimension of a module over a Weyl algebra in terms of a geometrical invariant of its characteristic variety, the involutive dimension. This is followed by some examples which show that this inequality may be strict.

1. INTRODUCTION

In this paper we show that the Krull dimension of a module over the Weyl algebra is related to geometrical invariants of the characteristic variety of that module. We begin with a survey of a few basic facts about modules over the Weyl algebra and their characteristic varieties.

Let A_n denote the n th Weyl algebra over the field of complex numbers \mathbb{C} . The generators of the algebra A_n will be denoted by x_i and $\partial_i = \partial/\partial x_i$ for $1 \leq i \leq n$. We endow A_n with the Bernstein filtration, obtained by giving each x_i and each ∂_i degree one. The k th step in this filtration will be denoted by $A_n(k)$. We write S_n for the graded ring of A_n with respect to the Bernstein filtration and $S_n(k)$ for its k th homogeneous component. The symbol of order k is the canonical map $\sigma_k: A_n(k) \rightarrow A_n(k)/A_n(k-1) = S_n(k)$. Thus S_n is a polynomial ring in the $2n$ variables y_1, \dots, y_{2n} , where $y_i = \sigma_1(x_i)$ and $y_{i+n} = \sigma_1(\partial_i)$ for $1 \leq i \leq n$.

Let $\{ , \}$ denote the Poisson bracket of S_n associated to the standard symplectic structure of \mathbb{C}^{2n} . The Poisson bracket is a bilinear and skew symmetric operation in S_n which satisfies the Leibniz rule: if $f, g, h \in S_n$, then $\{f, gh\} = g\{f, h\} + h\{f, g\}$. It may be used to calculate the symbol of a bracket of two operators in A_n as follows: If $d \in A_n(k)$ and $d' \in A_n(m)$, then $\sigma_{k+m-2}([d, d']) = \{\sigma_k(d), \sigma_m(d')\}$. For more details see [1] or [10].

Let M be a finitely generated left A_n -module and F a good filtration of M . By that we mean a filtration of M for which $\text{gr}^F M$ is finitely generated over S_n . Let $I(M)$ be the radical of the annihilator of $\text{gr}^F M$ in S_n . Note that $I(M)$ is a homogeneous ideal of S_n . The ideal $I(M)$ is called the *characteristic ideal* of M and its variety $\text{char}(M)$ in \mathbb{C}^{2n} , the *characteristic variety* of M . Both $I(M)$ and $\text{char}(M)$ are independent of the choice of the good filtration

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used to define them and hence, are invariants of M . The dimension $d(M)$ of $\text{char}(M)$ equals the Gelfand-Kirillov dimension of M .

The ideal $I(M)$ is an involutive ideal of S_n with respect to the Poisson bracket defined above, namely, $\{I(M), I(M)\} \subseteq I(M)$. Equivalently, the variety $\text{char}(M)$ is involutive in the sense that its tangent spaces are co-isotropic for the standard skew-symmetric form in \mathbb{C}^{2n} ; see [1]. The involutivity of $I(M)$ was proved by Gabber in [3] using purely algebraic methods. An important consequence of the involutivity of $I(M)$ is the inequality $d(M) \geq n$.

Let us now review some of the results that relate invariants of the characteristic variety and the Krull dimension of A_n -modules. Let M be a finitely generated left A_n -module, with Krull dimension $\text{Kdim}(M)$. It follows from [8, Corollary 8.5.6] that $d(M) \geq \text{Kdim}(M) + n$. Björk had conjectured in [2] that the equality holds in the formula above. This, however, is false. Stafford showed in [9] that there exist simple A_n -modules of Gelfand-Kirillov dimension $2n - 1$; his example is also discussed in [5, Chapter 8].

Following a suggestion of J. T. Stafford we consider in this paper another formula for the Krull dimension of modules over the Weyl algebra. This is inspired by results of Bernstein and Lunts in [1] and [7]. For a finitely generated left A_n -module M , let $\beta(M)$ be the length of the longest chain of involutive homogeneous irreducible subvarieties contained in $\text{char}(M)$. We prove in §2 that $\text{Kdim}(M) \leq \beta(M)$ and that $d(M) \geq \beta(M) + n$. However, both these inequalities may be strict, as shown by the examples of §3.

2. INVOLUTIVE DIMENSION

We begin with the definition of a new dimension for ideals of S_n . Let I be a homogeneous ideal of S_n . The *involutive dimension* of I , denoted by $\beta(I)$, is the length of the longest chain of homogeneous involutive prime ideals containing I . Let M be a finitely generated left A_n -module. It is convenient to define the *involutive dimension of M* , denoted by $\beta(M)$, to be the involutive dimension of $I(M)$; namely $\beta(M) = \beta(I(M))$. This is equivalent to the definition in terms of $\text{char}(M)$ given in the introduction. We proceed to state some of the properties of the involutive dimension. Our first lemma is a well-known result; we include a proof for the sake of completeness.

Lemma 2.1. *Let I be a homogeneous involutive radical ideal of S_n . The primes minimal over I are homogeneous involutive prime ideals of S_n .*

Proof. Let P be a prime ideal minimal over I . It is easy to show that P must be homogeneous; we show that it is involutive. Let $a, a' \in P$; we shall prove that $\{a, a'\} \in P$. Let J be the intersection of the prime ideals minimal over I and different from P . Let $b \in J \setminus P$. Since $JP \subseteq I$, we must have that $\{ab, a'b\} \in I \subseteq P$, but $\{ab, a'b\} = \{ab, a'\}b + \{ab, b\}a'$. Thus $\{ab, a'\}b \in P$. Since P is prime and $b \notin P$, one has that $\{ab, a'\} \in P$. Once again, $\{ab, a'\} = \{a, a'\}b + \{b, a'\}a$, from which we conclude that $\{a, a'\} \in P$, as required.

Recall that the *coheight* of an ideal I of S_n is the length of the longest chain of prime ideals of S_n that contain I . It will be denoted by $\text{coht}(I)$.

Lemma 2.2. *If I is an involutive ideal of S_n , then $\beta(I) \leq \text{coht}(I) - n$.*

Proof. Let $P_0 \subset P_1 \subset \dots \subset P_s$ be the longest chain of involutive homogeneous prime ideals containing I . Then $\text{coht}(I) \geq \beta(I) + \text{coht}(P_s)$. Since P_s is an involutive ideal, $\text{coht}(P_s) \geq n$. Hence $\text{coht}(I) \geq \beta(I) + n$.

Let J be a left ideal of A_n . The *symbol ideal* of J is

$$\sigma(J) = \sum_{k \geq 0} (\sigma_k(J \cap A_n(k))).$$

It is an ideal of S_n . The next result collects some of the properties of the involutive dimension. In particular it follows from Proposition 2.3(b) that β is an exact dimension function in the sense of [8, 6.8.4]. On the other hand, Proposition 2.3(d) is a generalisation of [1, Theorem A].

Proposition 2.3. *The involutive dimension of a finitely generated left A_n -module M satisfies*

- (a) $\beta(M) = \max\{\beta(P), P \text{ a prime minimal over } I(M)\}$.
- (b) *If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of finitely generated A_n -modules, then $\beta(M) = \max\{\beta(M'), \beta(M'')\}$.*
- (c) $d(M) \geq \beta(M) + n$.
- (d) *If J is a left ideal of A_n such that $\sigma(J)$ is prime and $\beta(\sigma(J)) = 0$, then J is a maximal left ideal of A_n . In other words, A_n/J is a simple left A_n -module.*

Proof. Note that (a) follows from Lemma 2.1 and the fact that $I(M)$ is a radical involutive ideal of S_n whilst (c) follows from Lemma 2.2. To prove (b), note that $I(M) = I(M') \cap I(M'')$. Thus, if P is a prime ideal of S_n , then it contains $I(M)$ if and only if it contains either $I(M')$ or $I(M'')$. The equality follows immediately from this and (a).

Finally, we prove (d). Suppose that L is a left ideal of A_n that properly contains J . Thus $\sigma(J) \subset \sigma(L) \subseteq S_n$. Taking radicals and recalling that $\sigma(J)$ is prime, one obtains the proper inclusion, $\sigma(J) \subset \text{rad}(\sigma(L))$. But $\text{rad}(\sigma(L)) = I(A_n/L)$ is an involutive homogeneous radical ideal of S_n . Since $\beta(\sigma(J)) = 0$, we must have that $\beta(\text{rad}(\sigma(L))) = 0$. But $\sigma(J)$ is prime and the inequality $\sigma(J) \subset \text{rad}(\sigma(L))$ is proper. Hence $\text{rad}(\sigma(L)) = S_n$ and $L = A_n$.

Now we shall prepare the ground for the proof that the involutive dimension is an upper bound for the Krull dimension. The main step in the proof of this result is Theorem 2.6. Its proof makes use of the multiplicity of an A_n -module at a prime ideal of S_n , which we briefly describe. Let M be a finitely generated left A_n -module with a good filtration F and P a prime ideal of S_n . The *multiplicity* $m_P(M)$ of M with respect to P is the length of the $(S_n)_P$ -module $(\text{gr}^F M)_P$. The multiplicity thus defined is independent of the good filtration F used to calculate it and is additive over short exact sequences of A_n -modules. The multiplicity also satisfies the following property: $m_P(M)$ is finite and nonzero if and only if $M \neq 0$ and P is a prime ideal minimal over $I(M)$. For details see [1, §3] and [6, Chapter V, Proposition 2.7]. In the proof of the theorem we use that, under certain circumstances, the length of a descending chain of A_n -modules is bounded by the multiplicity. This is the content of the following lemma. We omit the proof, as it is easy and very similar to [5, Corollary 7.8].

Lemma 2.4. *Let M be a finitely generated left A_n -module and P a prime minimal over $I(M)$. Suppose that $M = M_0 \supset M_1 \supset \dots \supset M_s$ is a strictly descending chain of submodules of M for which P is minimal over $I(M_i/M_{i+1})$ for $i = 0, 1, 2, \dots$. Then*

- (a) $m_P(M/M_i) = \sum_{j=0}^{i-1} m_P(M_j/M_{j+1})$,
- (b) $s \leq m_P(M)$.

One more technical lemma is necessary before we come to the proof of the theorem.

Lemma 2.5. *Let M be a finitely generated left A_n -module and N a submodule of M . If P is a homogeneous prime ideal of S_n minimal over $I(N)$ and P does not contain $I(M/N)$, then P is minimal over $I(M)$.*

Proof. Since $I(M) = I(N) \cap I(M/N)$, it follows that P contains $I(M)$. Suppose that P is not minimal over $I(M)$. Thus there exists a homogeneous prime ideal Q such that $I(M) \subseteq Q \subset P$. In particular $I(N) \cdot I(M/N) \subseteq Q$. Since $I(M/N)$ is not contained in P , we must have that $I(N) \subseteq Q$. But this contradicts the fact that P is minimal over $I(N)$.

We want to show that the Krull dimension of a module is bounded above by its involutive dimension. Following [8, Chapter 8], it is enough to prove that A_n is finitely partitive for β . Recall that A_n is (left) *finitely partitive* for β if given any finitely generated left A_n -module M , then every chain $M = M_0 \supseteq M_1 \supseteq \dots$ of submodules of M , such that $\beta(M_i/M_{i+1}) = \beta(M)$ for all $i \geq 0$, must terminate. This is the weaker definition of finitely partitive, proposed in [8, 8.7.3].

Theorem 2.6. *A_n is finitely partitive for the involutive dimension.*

Proof. Let M be a finitely generated left A_n -module. Suppose that $M = M_0 \supseteq M_1 \supseteq \dots$ is a chain of submodules of M such that $\beta(M_i/M_{i+1}) = \beta(M)$ for $i = 0, 1, 2, \dots$.

Let \mathbb{P} be the (finite) set of homogeneous involutive prime ideals minimal over $I(M)$. Given $i \in \mathbb{N}$ we have that $I(M_i/M_{i+1}) \supseteq I(M)$ and $\beta(M_i/M_{i+1}) = \beta(M)$; thus at least one prime ideal in \mathbb{P} must be minimal over $I(M_i/M_{i+1})$. Since \mathbb{P} is finite, there exists a prime $P \in \mathbb{P}$ which is minimal over infinitely many of the ideals $I(M_i/M_{i+1})$.

Now if P is minimal over $I(M_i/M_{i+1})$ but not over $I(M_{i-1}/M_i)$, then by Lemma 2.5 it is minimal over $I(M_{i-1}/M_{i+1})$. Thus by deleting from the sequence $M = M_0 \supset M_1 \supset \dots$ the terms M_i whenever P is not minimal over $I(M_{i-1}/M_i)$, we obtain an infinite subsequence

$$M = N_0 \supset N_1 \supset N_2 \supset \dots$$

for which P is minimal over all $I(N_i/N_{i+1})$. But this contradicts Lemma 2.4 and completes the proof of the theorem.

Corollary 2.7. *The involutive dimension of a finitely generated left A_n -module M satisfies:*

- (a) $d(M) \geq \beta(M) + n \geq \text{Kdim}(M) + n$.
- (b) *If $\beta(M) = 0$, then M is a module of finite length.*

Proof. The first inequality in (a) follows from Proposition 2.3(c). The second inequality follows from the fact that A_n is finitely positive for β (see [8, 8.7.3 and 8.3.18]), whilst (b) follows from (a).

Note that the inequalities above may be strict. This is illustrated in the next section, where several examples are discussed. It is perhaps worth observing that most of the above can be generalized to almost commutative rings.

3. EXAMPLES

We now discuss some examples for which the inequalities of the previous section are strict. Consider first $d(M) \geq \beta(M) + n$. Bernstein and Lunts showed in [1, 7] that for most operators $d \in A_n$, $n \geq 2$, the module $M = A_n/A_n d$ satisfies $\beta(M) = 0$. Since $d(M) = 2n - 1$, the inequality is strict in this case.

We now turn to $\text{Kdim}(M) \leq \beta(M)$. Let $d = \partial_1 + (1 + \lambda x_1 x_2) \partial_2 + x_2$. Stafford showed in [9] that the module $M = A_2/A_2 d$ is simple. Thus $\text{Kdim}(M) = 0$. But $I(M)$ is the ideal generated by $y_1 y_2 y_4$ in S_2 . Hence (y_4) is a prime ideal minimal over $I(M)$. But $(y_4) \subseteq (y_3, y_4)$ and (y_3, y_4) is involutive. Hence, $\beta(M) = 1$, and we conclude that $\text{Kdim}(M) < \beta(M)$, in this case.

The characteristic ideal of M , in this last example, is not prime. However, even when the characteristic ideal is prime, the inequality $\text{Kdim}(M) \leq \beta(M)$ may be strict. To construct such an example, we shall use a result of Bernstein and Lunts [1]. Let V_k be the space of vector fields on \mathbb{C}^2 of the form $P_1 \partial_1 + P_2 \partial_2$, where P_1 and P_2 are polynomials of degree $\leq k - 2$ in x_1, x_2 . For a vector field ξ in \mathbb{C}^2 and a singular point s of ξ , let $\Lambda(\xi, s)$ denote the subgroup of \mathbb{C} generated by the eigenvalues of the linear operator on $T_s \mathbb{C}^2$, defined by the one-jet of ξ .

Given a vector field $\xi = P_1 \partial_1 + P_2 \partial_2$, let Y_ξ be the hypersurface $P_1(y_1, y_2)y_3 + P_2(y_1, y_2)y_4 = 0$ of \mathbb{C}^4 . Thus Y_ξ is involutive with respect to the standard symplectic structure of \mathbb{C}^4 . It is also *conical*, that is, homogeneous with respect to the variables y_3 and y_4 . We say that Y_ξ is *minimal* if the only involutive conical variety contained in Y_ξ are of the following type: $Z(y_3, y_4)$ or $Z(y_1 - \alpha_1, y_2 - \alpha_2)$, for a point $(\alpha_1, \alpha_2) \in \mathbb{C}^2$. In other words, \mathbb{C}^4 is the cotangent bundle of \mathbb{C}^2 , and Y_ξ is minimal if its only involutive conical subvarieties are the zero section or fibres of this bundle.

Theorem 3.1. *Let $k \geq 4$ and $\xi \in V_k$. Suppose that Y_ξ is minimal and that ξ has a finite, nonempty set of singular points in \mathbb{C}^2 . Choose a polynomial $f \in \mathbb{C}[x_1, x_2]$ such that, for each singular point $s \in \mathbb{C}^2$, $f(s) \notin \Lambda(\xi, s)$. Then the operator $\xi + f$ generates a maximal left ideal of A_2 .*

Before this theorem is put to use, we must obtain vector fields ξ for which Y_ξ is minimal. First of all, we endow V_k with the topology of affine space. The natural structure follows from the fact that V_k is a complex vector space with basis $x_1^r x_2^s \partial_i$, where $r + s \leq k - 2$ and $i = 1, 2$. The corresponding coordinate functions will be denoted by z_{rsi} . We say that a property Σ holds *generically* in V_k if the set $\{\xi \in V_k : \Sigma \text{ does not hold for } \xi\}$ is contained in the union of a countable number of hypersurfaces in V_k . The next result is a consequence of [1, Theorem 5].

Theorem 3.2. *The property “ Y_ξ is minimal” holds generically in V_k for $k \geq 4$.*

The example is constructed as follows.

Theorem 3.3. *Let $4 \leq k \leq 7$ be an integer. There exists $d \in A_2$ which satisfies:*

- (a) A_2d is a maximal left ideal of A_2 .
 - (b) d has degree $m = k - 1$ for the Bernstein filtration, and the hypersurface $\sigma_m(d) = 0$ in $\mathbb{P}^3(\mathbb{C})$ is an irreducible homogeneous subvariety of $\mathbb{P}^3(\mathbb{C})$.
 - (c) $\sigma_m(d)$ is contained in the ideal generated by y_1 and y_2 in S_2 .
- In particular, A_2/A_2d is a simple module, but $I(A_2/A_2d)$ is prime and has involutive dimension 1.*

It is better to isolate part of the proof in a lemma. If $P \in \mathbb{C}[x_1, x_2]$, denote by P^k the homogeneous component of degree k of P .

Lemma 3.4. *Suppose that $k \geq 4$. Let X be the set of all vector fields $P_1\partial_1 + P_2\partial_2$ in V_k such that at least one of the three conditions below is satisfied:*

- (I) $P_1^{k-2} \cdot P_2^{k-2} = 0$.
- (II) P_1 and P_2 have a common factor.
- (III) P_1^{k-2} and P_2^{k-2} have a common factor.

Then X is contained in a union of hypersurfaces of V_k .

Proof. Let $\xi = P_1\partial_1 + P_2\partial_2$ be a vector field in X . If ξ satisfies (I), then it is contained in the union of the hypersurfaces of equations $z_{k-2,0,i} = 0$ for $i = 1, 2$.

Suppose that P_1, P_2 have a factor in common. We may assume that $z_{k-2,0,i} \neq 0$ for $i = 1, 2$. Thus P_1 and P_2 are monic as polynomials in x_1 . If they have a common factor in $\mathbb{C}[x_1, x_2]$, then they have a common factor in $\mathbb{C}(x_2)[x_1]$. Hence the resultant $\text{Res}_{x_1}(P_1, P_2)$, with respect to x_1 , is identically zero. The coefficients of this resultant with respect to x_2 are polynomials in z_{rsi} . Equating one of these coefficients to zero we have a hypersurface that contains ξ . Thus ξ is contained in a union of hypersurfaces when (II) holds. One may deal with (III) in a similar way.

Proof of Theorem 3.3. By Theorem 3.2 and Lemma 3.4, there exists $\xi \in V_k$ such that Y_ξ is minimal and (I), (II), and (III) are not satisfied by ξ . Let S_ξ stand for the set of singular points of ξ . We show that S_ξ is finite and nonempty. Let $\xi = P_1\partial_1 + P_2\partial_2$, where P_1, P_2 are polynomials of degree $\leq k - 2$.

A singularity of ξ is a zero of the system $P_1 = P_2 = 0$. Write W_1 and W_2 for the curves of $\mathbb{P}^2(\mathbb{C})$ that correspond to the sets of zeros of the homogenizations of P_1 and P_2 , respectively. Then $W_1 \cap W_2 \neq \emptyset$. Let $w \in W_1 \cap W_2$. If w belongs to the line at infinity, then its homogeneous coordinates are $[\alpha : \beta : 0]$. Hence (α, β) is a zero of P_1^{k-2} and P_2^{k-2} . Since these polynomials are homogeneous, this implies that they have a common factor, contradicting (III). Thus w must be a finite point, and so $w \in S_\xi \neq \emptyset$.

Since P_1 and P_2 do not have a common factor, we may apply Bezout's Theorem and conclude that S_ξ contains at most $(k - 2)^2$ elements. We will use this to show that there exists a polynomial $Q \in \mathbb{C}[x_1, x_2]$ of degree $\leq k - 1$ which has the elements of S_ξ for zeros. This is the place in the proof where we need to have that $k \leq 7$.

Let $U(k)$ be the vector space of polynomials in $\mathbb{C}[x_1, x_2]$ of degree $\leq k - 1$. Put $S_\xi = \{s_1, \dots, s_n\}$. Consider the linear mapping $\phi: U(k) \rightarrow \mathbb{C}^n$ defined by $\phi(Q) = (Q(s_1), \dots, Q(s_n))$. Since $\dim U(k) = \binom{k+1}{2} = (k^2 + k)/2$, it follows

that $\dim U(k) > (k-2)^2 \geq n$ for $k \geq 7$. Thus ϕ has a nonzero kernel. Hence there exists $0 \neq Q \in U(k)$ such that $Q(s) = 0$ for every $s \in S_\xi$.

Now let $L \subseteq \mathbb{C}$ denote the union of the $\Lambda(\xi, s)$ for every $s \in S_\xi$. There are at most $2 \cdot (k-2)^2$ such eigenvalues. Choose a complex number α algebraically independent with the set L . Put $f = Q + \alpha$. For each $s \in S_\xi$ one has that $f(s) = Q(s) + \alpha = \alpha \notin \Lambda(\xi, s)$. Thus we may apply Theorem 3.1 to the operator $d = P_1\partial_1 + P_2\partial_2 + f$ and conclude that the left ideal A_2d is maximal. Hence (a) is verified.

Let $G = \sigma_{k-1}(d) = P_1^{k-2}(y_1, y_2)y_3 + P_2^{k-2}(y_1, y_2)y_4 + Q^{k-1}(y_1, y_2)$. Note that G is linear in y_3, y_4 . If it were reducible, then at least one of its factors would be a nonconstant polynomial in y_1, y_2 . But this is not possible, because P_1^{k-2} and P_2^{k-2} have no common factor. Thus G must be irreducible homogeneous polynomial. Therefore, the ideal (G) in S_2 is prime, and $G = 0$ defines an irreducible hypersurface in $\mathbb{P}^3(\mathbb{C})$.

That $G \subseteq (y_1, y_2)$ is immediate; thus (c) is also verified. Since (y_1, y_2) is obviously prime and involutive, it follows that $\beta(A_2/A_2d) \geq 1$. Thus, by Corollary 2.7, $\beta(A_2/A_2d) = 1$, and the proof is complete.

The fact that the inequality $\text{Kdim}(M) \leq \beta(M)$ may be strict admits a geometrical interpretation. It is closely related to the breakdown of the following extension property for left ideals of A_n . Let J be a left ideal of A_n . Suppose that P is a homogeneous prime ideal of height 1 over $\text{rad}(\sigma(J))$. Is there a left ideal L of A_n such that $J \subseteq L$ and P is minimal over $\text{rad}(\sigma(L))$? A counterexample to this extension property is provided by the ideal $A_n d$ of Theorem 3.3. In fact, if $J = A_n d$, then $\text{rad}(\sigma(J)) = (G)$ is a prime ideal. The prime ideal (y_1, y_2) has height 2 and contains (G) , which has height 1. Hence (y_1, y_2) has height 1 over (G) . But $J = A_n d$ is maximal; hence the extension property must fail in this case.

Let us end with a question related to the above extension property, the answer to which does not seem to be known. Let J be a radical involutive homogeneous ideal of S_n . Is there a finitely generated left A_n -module M such that $I(M) = J$? On this question, see also [10, Chapter I, Theorem 6.21] and [2, Chapter 4, Theorem 6.5].

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