ON THE DIFFERENTIAL SIMPLICITY OF AFFINE RINGS

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Abstract. We prove that every complex regular affine ring is differentially simple relative to a set with only two derivations.

The study of the differential simplicity of commutative rings has known a resurgence of interest in recent years, but its basic results go back at least to the 1950s. In order to review the results that seem most relevant to the theme of this note we introduce a few definitions.

Let K be a commutative ring. A derivation d of a (commutative) K-algebra R is an endomorphism of the additive group of R that satisfies $d(K) = 0$ and Leibniz's rule for the differentiation of a product, namely

$$d(ab) = ad(b) + bd(a) \text{ for all } \quad a, b \in R.$$

Denoting by $\mathrm{Der}_K(R)$ the set of all derivations of R, let $\emptyset \neq \mathcal{D} \subset \mathrm{Der}_K(R)$ be a family (finite or not) of derivations of R. An ideal I of R is $\mathcal{D}$-*stable* if $d(I) \subset I$ for all $d \in \mathcal{D}$. Of course the ideals 0 and R are always $\mathcal{D}$-stable. If R has no other $\mathcal{D}$-stable ideal it is called $\mathcal{D}$-*simple* (or *differentially simple* if $\mathcal{D} = \mathrm{Der}_K(R)$).

The 1960s saw a flurry of results on differentially simple rings, among them the classification of differentially simple algebras that are finite dimensional over a field [2] or that are affine over an algebraically closed field of positive characteristic [9]. From our point of view the most interesting result of that decade was Seidenberg's proof in [14, Theorem 3, p. 26] that every regular affine ring over a field of characteristic zero is differentially simple.

In the 1970s some of the most impressive results in the area concerned rings that are differentially simple relative to a one element family $\{d\}$. To simplify the notation these rings are called d-*simple* and the corresponding derivation d is said to be *simple*. In [10] R. Hart proved that the local ring at a nonsingular point of an irreducible variety over a field of characteristic zero is always d-simple. In the same paper he exhibited an example of a regular affine $\mathbb{Q}$-algebra R that is not d-simple, for any choice of $d \in \mathrm{Der}_K(R)$. On the other hand, George Bergman (unpublished) showed that the ring of polynomials $\mathbb{Q}[x, y]$ is d-simple for an appropriately chosen derivation d. More examples of simple derivations have appeared since then in [1], [13], [5], [3] and [6]. For some recent applications in noncommutative algebra see [4], [11] and [8].

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In this paper we return to the consideration of differential simplicity relative to families of more than one element. It follows from the result of Seidenberg mentioned above that an affine algebra over an algebraically closed field is always differentially simple with respect to a *finite* family $\mathcal{D}$ of derivations. For example, we may take $\mathcal{D}$ to be a set of generators for $\mathrm{Der}_K(R)$. Let us call a ring *k-differentially simple* if it is differentially simple relative to a family with k derivations. In particular, if $\mathrm{Der}_K(R)$ is a free R-module then R is $\dim(R)$-differentially simple. However, this is not the smallest k for which such a ring can be k-differentially simple. Indeed, as proved in [1] a polynomial ring with n variables over a field of characteristic zero is always d-simple, hence 1-differentially simple, whilst its Krull dimension is n. Thus, one may ask: given a differentially simple ring R, what is the smallest positive integer k such that R is k-differentially simple?

Theorem. *If X is a smooth complex irreducible affine variety then $\mathcal{O}(X)$ is 2-differentially simple.*

The proof of the theorem depends on a few lemmas, that we state on a slightly more general context than required by the theorem itself.

Lemma 1. *Let $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ be distinct maximal ideals of a regular affine domain $\mathcal{O}$ defined over a field K of characteristic zero. There exists a derivation $d \in \mathrm{Der}_K(\mathcal{O})$ such that $d(\mathfrak{m}_j) \not\subset \mathfrak{m}_j$ for every $1 \leq j \leq t$.*

Proof. By [14, Theorem 3, p. 26] there exist derivations $\delta_1, \dots, \delta_t \in \mathrm{Der}_K(\mathcal{O})$ such that $\delta_j(\mathfrak{m}_j) \not\subset \mathfrak{m}_j$, for every $1 \leq j \leq t$. For each $1 \leq i \leq t$, choose

$$\alpha_i \in (\mathfrak{m}_1 \cap \dots \cap \widehat{\mathfrak{m}}_i \cap \dots \cap \mathfrak{m}_t) \setminus \mathfrak{m}_i,$$

where the hat means that the ideal is omitted from the intersection, and consider the derivation $d = \sum_{j=1}^t \alpha_j \delta_j$. Since $\alpha_j \in \mathfrak{m}_i$ whenever $j \neq i$, it follows that

$$d(\mathfrak{m}_i) \subset \mathfrak{m}_i \text{ if and only if } \alpha_i \delta_i(\mathfrak{m}_i) \subset \mathfrak{m}_i.$$

But, by the primality of $\mathfrak{m}_i$, this last inclusion occurs if and only if either $\alpha_i \in \mathfrak{m}_i$ or $\delta_i(\mathfrak{m}_i) \subset \mathfrak{m}_i$, both of which are false. $\square$

The proof of the theorem uses a result of the theory of holomorphic foliations that we briefly review. Given a smooth complex projective variety Y, a foliation on Y is a locally free $\mathcal{O}_Y$-submodule of the tangent sheaf Θ_Y. These modules correspond to maps $\mathcal{L}^{-1} \to \Theta_Y$, where $\mathcal{L}$ is an invertible bundle over Y. Therefore, the space that parametrizes all the foliations for a fixed line bundle $\mathcal{L}$ is $\mathrm{H}^0(Y, \mathcal{L} \otimes \Theta_Y)$.The result we require is an immediate consequence of [7, Theorem 1.1, p. 118].

Lemma 2. *Let Y be a smooth complex projective variety Y and let $\mathcal{L}$ be an ample line bundle over Y. For all $k \gg 0$ there are foliations in $\mathrm{H}^0(Y, \mathcal{L}^{\otimes k} \otimes \Theta_Y)$ whose only invariant proper algebraic subvarieties are finitely many points.*

Recall that a subvariety Z of Y is invariant under a foliation $\theta \in \mathrm{H}^0(Y, \mathcal{L} \otimes \Theta_Y)$ if the ideal $I(Z \cap U)$ is stable under the section $\theta(U)$ for every affine open set U of Y.

Lemma 3. *Let $\pi : \widetilde{X} \to X$ be a projective map of projective irreducible varieties over an algebraically closed field of characteristic zero. Suppose that U is an affine open set of X and let $\widetilde{U} = \pi^{-1}(U)$. If $\pi : \widetilde{U} \to U$ is an isomorphism, then there exists an ample invertible sheaf $\mathcal{L}$ over $\widetilde{X}$ whose restriction to $\widetilde{U}$ is a free $\mathcal{O}_{\widetilde{U}}$-module.*

Proof. By [12, Theorem 1.24, p. 328] the birational map π is a blowing-up at some closed scheme $Z \subset X \setminus U$. Let $\mathcal{I}$ be the ideal sheaf of Z in $\mathcal{O}_X$ and let $\mathcal{L} = \mathcal{I}\mathcal{O}_{\widetilde{X}}$. By [12, Theorem 1.22, p. 327], $\mathcal{L}$ is a very ample sheaf with respect to π, and

$$\mathcal{L}_{\widetilde{U}} \cong (\mathcal{I}\mathcal{O}_{\widetilde{X}})_{\widetilde{U}} \cong O_{\widetilde{U}}.$$

Thus $\mathcal{L}$ is the required ample invertible sheaf. □

Proof of the Theorem. Let $\overline{X}$ be the closure of X in $\mathbb{P}^n$. Since X is smooth by hypothesis, it follows that the singular locus S of $\overline{X}$ is contained in the hyperplane at infinity H_∞ of $\mathbb{P}^n$. By Hironaka's Theorem on the resolution of singularities, $\overline{X}$ admits a projective desingularisation Y. From Lemma 3 we know that there exists an ample line bundle $\mathcal{L}$ on Y such that $\mathcal{L}|_X$ is a free $\mathcal{O}_X$ module. By Lemma 2, for every $k \gg 0$, there exists a foliation

$$\theta \in \mathrm{H}^0(Y, \Theta_Y \otimes \mathcal{L}^{\otimes k})$$

whose only proper invariant algebraic subvarieties are its finitely many singular points. Since X is affine and $\mathcal{L}|_X$ is a free $\mathcal{O}_X$-module of rank one, this foliation is generated over X by a single derivation δ. Denote by M the set of maximal ideals of the coordinate ring $\mathcal{O}(X)$ that correspond, under the Nullstellensatz, to the singularities of δ. By Lemma 1, there exists a derivation $d \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}(X))$ under which none of these ideals is invariant. Take $T = \{\delta, d\}$. By [14, Theorem 1, p. 24] the result will follow if we prove that there are no T-stable prime ideals. So, let P be such an ideal. In order to be T-stable, P must satisfy $\delta(P) \subset P$. This implies that $P \in M$, so it cannot be stable under d, completing the proof of the theorem. □

We end the paper with a few questions. The first two inevitably deal with weakening the hypotheses of the theorem. Does the theorem remain true:

(1) if the base field is not $\mathbb{C}$?
(2) if the affine ring is replaced by a regular notherian ring?

Although the results of [7] require the base field to be $\mathbb{C}$, the theorem probably holds over any algebraically closed field. Non algebraically closed fields pose a more serious problem. Finally,

(3) what is the smallest k relative to which $\mathcal{O}(X)$ is k-differentially simple if the families of derivations with respect to which k is being computed are required to be Lie subalgebras of $\mathrm{Der}_K(X)$?

This last problem is clearly the one we must solve if our interest is on applications of these families in noncommutative algebra.

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